A Windblown Sea

What is wave turbulence?

Wave Turbulence is the study of the long time statistical behaviour of solutions of nonlinear field equations, usually conservative and Hamiltonian, describing a sea of weakly nonlinear interacting dispersive waves. In most interesting contexts, the system is nonisolated having both sources and sinks of energy and other conserved densities.

Why is it believed to be a “solved” problem?

- A natural asymptotic closure
- A closed kinetic equation for the particle or wave action density n_k from which all other quantities of interest such as *nonlinear frequency modification, higher order cumulants, spatial structure functions* can be calculated.
- In addition to the usual thermodynamic stationary solutions, the kinetic equation has finite flux (Kolmogorov-Zakharov) solutions which capture the flow of conserved densities (energy, waveaction) from sources to sinks.

But, it is not a “solved” problem and the story is far from over

- The KZ solutions are almost never uniformly valid in *k* space as we see with
  - Whitecaps
  - Filamentation of optical waves
- In some cases, often one dimensional situations, the KZ solutions are not valid at any *k* and the WT closure fails.
  - FPU, self induced transparency, MMT

A course on Wave Turbulence

- Closure: how and why it is achieved.
- Analytics: Asymptotic behaviors of integrals such as

$$\int f(\vec{k},\vec{k}_1,\vec{k}_2)\,\frac{e^{i(s_1\omega(k_1)+s_2\omega(k_2)-s\omega(k))t}-1}{i(s_1\omega_1+s_2\omega_2-s\omega)}\,\delta(k_1+k_2-k)\,dk_1\,dk_2$$

The nature of the zeros of $h = s_1\omega_1+s_2\omega_2-s\omega$

- Statistically steady states and conserved densities.

  Thermodynamic + Kolmogorov-Zakharov (KZ) spectra

- How KZ spectra are achieved in infinite and finite* capacity cases.

  * Anomolous behavior

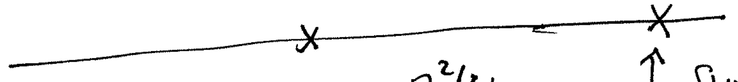
- Breakdown of KZ solutions in ranges of k space

  Asymptotic expansions and ratios $\frac{t_L}{t_{NL}}$ which are k dependent and are no longer small.

A course on Wave Turbulence 2.

- What one does when breakdown occurs

Two scenarios:



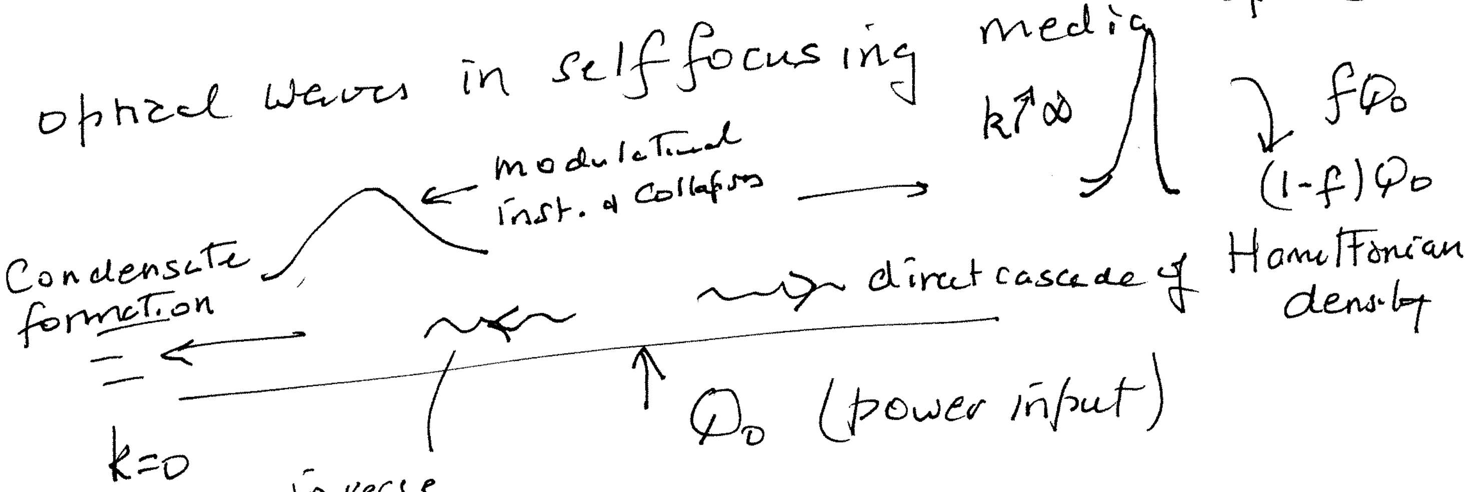
$(k_B = \frac{p^{2/3}k}{g}$

↑ surface tension sink

$k_{\text{cap.}} \sim \sqrt{\frac{\rho g}{S}}$

a. gravity wave

b. optical waves in self focusing media



Steady state

$$\Phi_0 + \Phi_0(1-f) + \Phi_0(1-f)^2 + \dots = \frac{\Phi_0}{f}$$

- Breakdown of spatial homogeneity and the influence of coherent structures

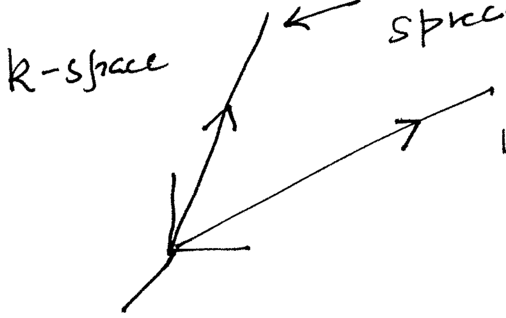
— Just because ε is small does not mean there are no fully nonlinear structures

FPU, SIT, MMT

and their resolution.

A course on Wave Turbulence 3.

- Second closures as in the cases of acoustic and shallow water ($\omega = c|\vec{k}|$) waves.



1st closure describes spread along resonant manifolds which are rays in kspace

"resonant manifold broadening"

Question: How do the diffrent rays talk to each other?

⇒ "Were the dinosaurs frozen or fried?" ⇐

MHD waves e.g. Alfven $\omega = \vec{H}_0 \vec{k}$

How come then is closure at all?

PREMISES

Premise 1 (P1): First, we assume the fields are spatially homogeneous and that ensemble averages of fields evaluated at the set of points $\mathrm{x}, \mathrm{x}+\mathrm{r}_1, \mathrm{x}+\mathrm{r}_2, \cdots$ depend only on the separations $\mathrm{r}_1, \mathrm{r}_2, \cdots$.

Premise 2 (P2): Second, we assume that at some initial point in time, the moment at which the external driving, e.g. the storm over the sea surface, is initiated, the fields at distant points are uncorrected. This means that the physical space cumulants have the property that, as the separations $|\mathbf{r}_j|$ become large, the cumulants decay sufficiently rapidly that their Fourier transforms are ordinary functions. This is a mild assumption but it is necessary because in the evaluation of the long time behavior of integrals such as $\int f(x)\frac{\sin xt}{x}dx$ need to know that $f(x)$ is sufficiently smooth in wavenumber x so that this integral behaves in long time as $\pi \mathrm{sgn}(t) f(0)$.

PREMISES

Premise 3 (P3): Third, we must ensure that various asymptotic expansions for the slow evolution of such two point functions as the waveaction density $n_{\mathbf{k}}$ remain uniformly valid in wavenumber. In its simplest form this means that the ratio of linear (t_L) to nonlinear (t_{NL}) time scales is small at all wavenumbers. It also means that all asymptotic expansions for the slow evolution of the waveaction density $dn_{\mathbf{k}}/dt$, the frequency renormalization which accounts for the slow time behavior of the leading order, higher order cumulants, and for the structure functions remain uniformly valid in wavenumber on almost all relevant solutions. The reason that we require $\frac{t_L}{t_{NL}}(k) \ll 1$ (with $k = |k|$) is that when we look for the long time behavior of integrals such as $\int f(x)\frac{\sin xt}{x}dx$ we want to know that the multiplying function $f(x)$ which will contain products of the waveaction densities $n_{\mathbf{k}}$ is not only smooth in x (wavenumber) but also that it varies slowly in time.

PREMISES

Premise 4 (P4): This premise says that one must test the deterministic theory first. If the field remains asymptotically linear (which might be tested by numerical simulations), we might surmise that this would rule out the appearance of coherent structures also dominating the long time behavior of the random system. The thinking here is that the deterministic problem would rule out resonances creating secular behavior (because the wavepackets are finite in length and so resonances do not produce long time cumulative effects) but not the appearance of coherent structures. If the latter do not appear in the deterministic system, the argument is that they will play no role in statistical ensembles either.

Premise 5 (P5): All KZ solutions are stable against perturbations which spontaneously break the spatial homogeneity symmetry.

The theory in outline

The variables

$$u^s(\mathbf{x}, t)(e.g.\ \eta(\mathbf{x}, t), \varphi(\mathbf{x}, z = \eta(\mathbf{x}, t), t)) \leftrightarrow A^s_{\mathbf{k}};\ \ \dim \mathbf{k} = d.$$

The governing equations

$$\frac{dA^s_{\mathbf{k}}}{dt} - is\omega_{\mathbf{k}} A^s_{\mathbf{k}} =$$

$$\textstyle\sum_{r=2}^{\infty} \varepsilon^{r-1} \int L^{ss_1\ldots s_r}_{kk_1\ldots k_r} A^{s_1}_{k_1} \ldots A^{s_r}_{k_r} \delta(k_1 + \cdots + k_r - k) dk_1 \ldots dk_r.$$

The theory in outline

The statistics

$< A_{\mathbf{k}}^{s} A_{\mathbf{k}'}^{-s} >= \delta(\mathbf{k}+\mathbf{k}') N_{\mathbf{k}}^{s}$ 2 point correlation

$\{A_{\mathbf{k}}^{s} A_{\mathbf{k}'}^{s'} \ldots\}$ higher order cumulants

The theory in outline 2

The strategy

1. Form BBGKY hierarchy.
2. Solve iteratively in power series in ε.
3. Choose "slow" variations of leading approximations to make 2. uniformly valid for "long" times.

The outcome $(N_k(t) = n_k(t) + \text{corrections} + \cdots)$

$$\frac{dn_k}{dt} = \varepsilon^2 T_2[n_k] + \varepsilon^4 T_4[n_k] + \varepsilon^6 T_6[n_k] + \cdots$$

$$s\omega_k \longrightarrow s\omega_k + \varepsilon^2 \Omega_2^s[n_k] + \cdots$$

Closure: How and why. Pg 1.

BBGKY hierarchy

$$\frac{d\varphi^{(2)}}{dt} = \varepsilon \int \varphi^{(3)} + \text{-----}$$

linear propagation

$$\frac{d\varphi^{(3)}}{dt} \mp i(s_1\omega(k_1) + s_2\omega(k_2) + s_3\omega_3(k_3))\,\varphi^{(3)}$$

$$= \varepsilon \int \varphi^{(4)}, \quad \varphi^{(2)}\varphi^{(2)}$$

nonlinear regeneration

To leading order: Linear wave propagation

$$\langle u u u \rangle \sim \int \varphi_0^{(3)}\, e^{i(\omega(k_1)+\omega(k_2)+\omega(k_3))t} \;\text{--}\; \xrightarrow{t\to\infty} 0$$

by RL

$\varphi_0^{(3)}$ smooth if $R^{(3)}(r_1, r_2, t=0)$ decays $r_1, r_2 \to \infty$.

$t \gg 1$

$\longrightarrow$ joint Gaussian state

εt finite Cumulants $(n \geq 3) \longrightarrow 0$

Closure: How and why, Pg 2.

At longer times, regeneration of higher order cumulants

$\varphi_1^{(3)}$ (order ε component of third order cumulant)

is regenerated by

$$\int \varphi_0^{(4)} \quad \text{and} \quad \int \varphi_0^{(2)} \varphi_0^{(2)}$$

↑ This leads to bounded behavior in time & therefore plays no long time role

↑ this leads to unbounded behavior Also, the $O(\varepsilon)$ component of $\varphi^{(3)}$ becomes nonsmooth. due to resonances.

$$\frac{d\varphi_2^{(2)}}{dt} = \varepsilon \int \varphi_0^{(3)} + \varepsilon^2 \int \left\{ \begin{matrix} \varphi_1^{(3)} \\ \underbrace{\varphi_0^{4}}_{\text{bounded behavior}} + \underbrace{\varphi_0^{(2)} \varphi_0^{(2)}}_{\text{unbounded}} \end{matrix} \right.$$

⇒ Kinetic equation for $\varphi^{(2)}(n_k)$.

Closure : How and why : Pg 3.

Unbounded behavior in $\psi^{(3)}$ and higher

$$\frac{d\psi^{(3)}}{dt} + i\left(\omega(k_1)+\omega(k_2)+\omega(k_3)\right)\psi^{(3)}$$

$$= \varepsilon \int \psi_0^{(4)} + \psi^{(2)}\psi^{(2)}$$

$$+ \varepsilon^2 \int \psi_0^{(5)} + \underbrace{\psi_0^{(3)}\psi_0^{(2)}}$$

↓

unbounded behavior

$$i\psi_0^{(3)}\left(\Omega(k_1;n_k)+\Omega(k_2;n_k)+\Omega(k_3,n_k)\right)$$

Regeneration of higher order cumulants is DOMINATED by products of lower order ones !!!

Unbounded behavior on all <u>higher order cumulants</u> removed by frequency renormalization

$$\omega \longrightarrow \omega(k) + \varepsilon^2\,\Omega(k;\,n_k)$$

$$\int \cdots\; n_k\, dk$$

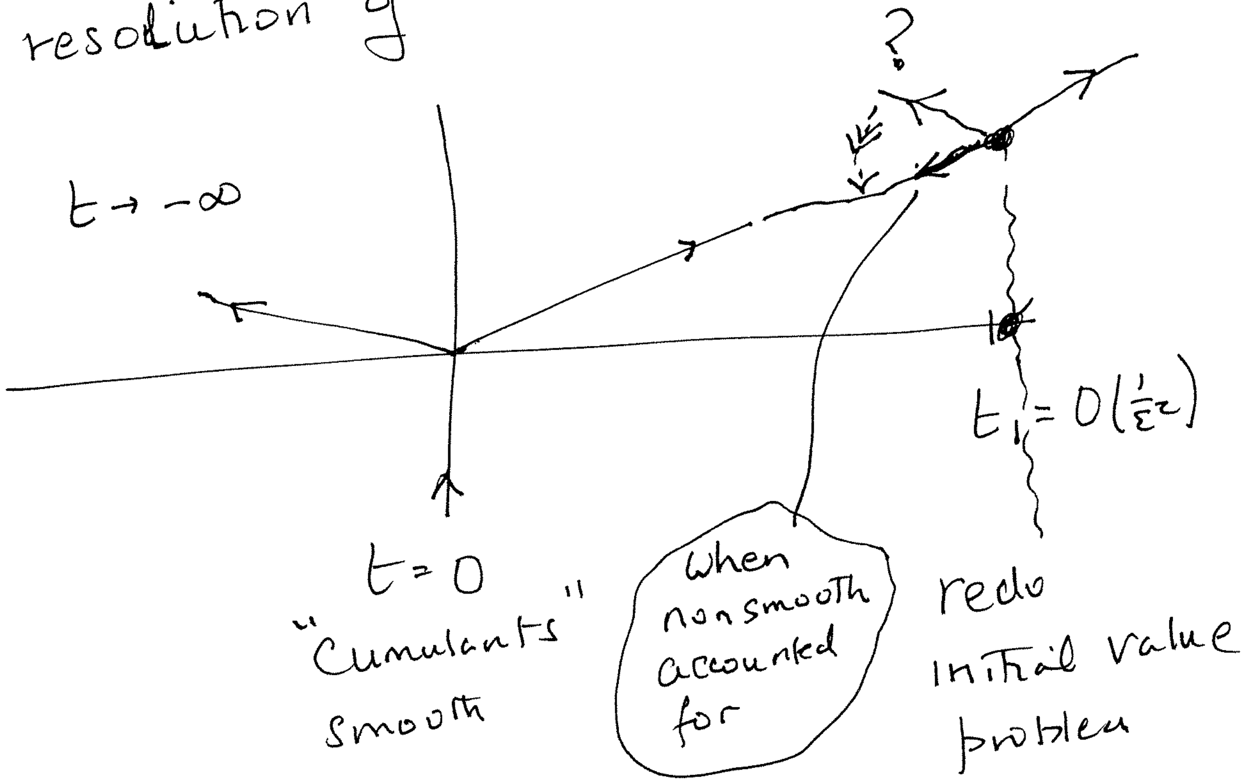
Closure : How and why

Kinetic eqn : $(\varphi^{(2)} \leftrightarrow n_k \text{ or } \omega_k n_k)$

$$\frac{dn_k}{dt} = \varepsilon^2 \int |L|^2 n n_1 n_2 \left(\frac{s_1}{n} - \frac{s_2}{n_1} - \frac{s}{n_2} \right) \delta(s_1\omega_1 + s_2\omega_2 - s\omega)\, \delta(\vec{k}_1 + \vec{k}_2 - \vec{k})\, d\vec{k}_1 d\vec{k}_2$$

factor (sgn t)

The resolution of the "irreversibility" paradox



Resolution: At t_1, the higher order cumulants are no longer smooth because of long distance resonant wave interactions. Corrections built by resonant wave interactions. Additional terms must be added and then "Hey presto"

Closure: How and Why? Pg 5.

"Random phase" approximation is NOT required.

It is *not true* that one can treat the phases of the higher order moment products

e.g. $<A\ A\ A>$ ~~as~~

as random.

Indeed we have seen $<A_0 A_0 A_1> \rightarrow \Phi_1^{(3)}$ develops δ function behavior.

What you can do — *but do not* *have to do* (and should not do if you want to calculate frequency renormalization) is treat products of zero order moments

$<A_0(k_1)\ A_0(k_2)\ A_0(k_3) --- >$ as if the phases of the A_0's were random.

The cumulant hierarchy: For the purposes of this tutorial, we follow, in detail, the time evolution of cumulants of order two and three, and indicate what happens to the hierarchy. In particular, we define

$$\begin{aligned}
\langle A^s_{\mathbf{k}} A^{s'}_{\mathbf{k}'}\rangle &= \delta(\mathbf{k}+\mathbf{k}')Q^{ss'}(\mathbf{k}'),\\
\langle A^s_{\mathbf{k}} A^{s'}_{\mathbf{k}'} A^{s''}_{\mathbf{k}''}\rangle &= \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')Q^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}''),\\
\langle A^s_{\mathbf{k}} A^{s'}_{\mathbf{k}'} A^{s''}_{\mathbf{k}''} A^{s'''}_{\mathbf{k}'''}\rangle &= \delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''')Q^{ss's''s'''}(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''') \qquad (1.2)\\
&\quad + P^{00'0''}\delta(\mathbf{k}+\mathbf{k}')\delta(\mathbf{k}''+\mathbf{k}''')\\
&\quad \times Q^{ss'}(\mathbf{k},\mathbf{k}')Q^{s''s'''}(\mathbf{k}'',\mathbf{k}'''),
\end{aligned}$$

where $P^{00'0''}$ stands for the cyclic permutation over $(s,\mathbf{k})$, $(s',\mathbf{k}')$, $(s'',\mathbf{k}'')$.

In writing down (1.2), we have assumed that (i) the mean $\langle u^s\rangle$ is identically zero for all times (the condition $L^{ss_1\ldots s_r}_{\mathbf{0}\mathbf{k}_1\ldots\mathbf{k}_r}=0$ in (1.1) guarantees that if the mean is initially zero, it stays zero) and (ii) the property of spatial homogeneity, namely that the ensemble average of products such as $R^{ss'}(\mathbf{r}) = \langle u^s(\mathbf{x})u^{s'}(\mathbf{x}+\mathbf{r})\rangle$ only depends on the relative geometry coordinate $\mathbf{r}$ and not on the base coordinate $\mathbf{x}$. This second property, which we have called premise P1, gives us that

$$\begin{aligned}
\langle A^s(\mathbf{k})A^{s'}(\mathbf{k}')\rangle &= \frac{1}{(2\pi)^{2d}}\int \langle u^s(\mathbf{x}_1)u^{s'}(\mathbf{x}_2)\rangle \mathrm{e}^{-i\mathbf{k}\mathbf{x}_1-i\mathbf{k}'\mathbf{x}_2}d\mathbf{x}_1 d\mathbf{x}_2\\
&= \frac{1}{(2\pi)^{2d}}\int \mathrm{e}^{-i(\mathbf{k}+\mathbf{k}')\mathbf{x}_1}d\mathbf{x}_1 R^{ss'}(\mathbf{r})\mathrm{e}^{-i\mathbf{k}'\mathbf{r}}d\mathbf{r}\\
&= \frac{\delta(\mathbf{k}+\mathbf{k}')}{(2\pi)^{d}}\int R^{ss'}(\mathbf{r})\mathrm{e}^{-i\mathbf{k}'\mathbf{r}}d\mathbf{r}\\
&= \delta(\mathbf{k}+\mathbf{k}')Q^{ss'}(\mathbf{k},\mathbf{k}'),
\end{aligned}$$

where $Q^{ss'}(\mathbf{k},\mathbf{k}')$ is the Fourier transform of $R^{ss'}(\mathbf{r})$ and $\mathbf{x}_2=\mathbf{x}_1+\mathbf{r}$. As we shall see, the delta function correlation of Fourier amplitudes is directly responsible for the nature of the long time secular behavior of iterates of the Fourier cumulants and thereby for the wave turbulence closure.

By multiplying (1.2) by $A^{s'}_{\mathbf{k}'}$ and the equivalent equation for $A^{s'}_{\mathbf{k}'}$ by $A^s_{\mathbf{k}}$ and adding and averaging, we obtain the first equation in the cumulant hierarchy,

$$\begin{aligned}
&\frac{dQ^{ss'}(\mathbf{k}')}{dt} - i(s\omega+s'\omega')Q^{ss'}(\mathbf{k}')\\
&\quad = \epsilon P^{00'}\sum_{s_1 s_2}\int L^{ss_1s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}Q^{s's_1s_2}(\mathbf{k}',\mathbf{k}_1,\mathbf{k}_2)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2, \qquad (1.3)
\end{aligned}$$

where $\mathbf{k} + \mathbf{k}' = 0$ and $\omega' = \omega(|\mathbf{k}'|)$; for $\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = \mathbf{0}$, by a similar calculation, we obtain the second equation in the cumulant hierarchy to be

$$\begin{aligned}\frac{dQ^{ss's''}(\mathbf{k}, \mathbf{k}'\mathbf{k}'')}{dt} &- i(s\omega + s'\omega' + s''\omega'')Q^{ss's''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'')\\ &= \epsilon P^{00'0''} \sum_{s_1 s_2} \int L^{ss_1 s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} Q^{s's''s_1 s_2}(\mathbf{k}', \mathbf{k}'', \mathbf{k}_1, \mathbf{k}_2)\\ &\quad \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2\\ &\quad + 2\epsilon P^{00'0''} \sum_{s_1 s_2} L^{ss_1 s_2}_{\mathbf{k}-\mathbf{k}'-\mathbf{k}''} Q^{s_1 s'}(\mathbf{k}') Q^{s_2 s''}(\mathbf{k}''). \end{aligned} \tag{1.4}$$

In (1.3) and (1.4), we observe that the time derivatives of cumulants depend on cumulants of higher order. In order to obtain the first closure, the first two equations in the hierarchy will be sufficient. We have also omitted writing down the terms arising from the cubic product of amplitudes in (1.1) but will restore them in the final answer.

The strategy for analyzing the cumulant hierarchy: So what is the strategy for solving (1.3), (1.4) and the other members of the cumulant hierarchy? On the surface, the equation hierarchy has the closure problem of fully developed turbulence. Equations for cumulants of order r involve equal, higher and lower order cumulants. However, when we solve the hierarchy iteratively by expanding each cumulant as an asymptotic expansion in powers of ϵ,

$$\begin{aligned}&Q^{ss'\ldots s^{N-1}}(\mathbf{k}, \mathbf{k}', \ldots \mathbf{k}^{(N-1)}, t)\\ &\quad = q_0^{ss'\cdots}(0)e^{i(s\omega + s'\omega' + \cdots)t} + \epsilon Q_1^{ss'\cdots} + \epsilon^2 Q_2^{ss'\cdots}, \end{aligned} \tag{1.5}$$

we find that a remarkable simplification occurs. In examining the long time behavior of the iterates Q_r, $r = 1, 2, \ldots$, we find some terms are bounded in t and some grow linearly in t. If the iterates of the Fourier space cumulants should contain generalized functions, then we simply ask that the corresponding expansions of the physical space cumulants are uniformly ordered in the limit of large t. The reason one gains closure is that all the unbounded terms arising in the iterates of the rth-order cumulant involve only zeroth-order cumulants $q_0^{ss'\cdots}$ of order r or less.

For example, simple calculations reveal that there are no secular terms in $Q_1^{ss'}(\mathbf{k}, \mathbf{k}')$ and that the secular terms in $Q_2^{ss'}(\mathbf{k}, \mathbf{k}')$, $s' = -s$ ($Q^{-ss}(\mathbf{k})$ is the waveaction density) are simply ($q_0^{-ss}(\mathbf{k})$ we call $n_{\mathbf{k}}$ and, for convenience,

$$\frac{dn_k}{dt} = 4\pi\varepsilon^2 \operatorname{sgn} t \sum_{s_1 s_2} \int L^{s s_1 s_2}_{k k_1 k_2} n_k n_{k_1} n_{k_2} \left(\frac{L^{-s-s_1-s_2}_{-k-k_1-k_2}}{n_k} + \frac{L^{s_1 s -s_2}_{k_1 k -k_2}}{n_{k_1}} + \frac{L^{s_2 s -s_1}_{k_2 k -k_1}}{n_{k_2}} \right) \delta(s_1\omega_1+s_2\omega_2-s\omega)\, \delta(\vec{k}_1+\vec{k}_2-\vec{k})\, d\vec{k}_1 d\vec{k}_2$$

$$\frac{d q_0^{s s' \ldots}(\vec{k}, \vec{k}', \ldots)}{dt} = i q_0^{s s' \ldots}(\vec{k}, \vec{k}' \ldots) \left(\Omega_2^{s}(k) + \Omega_2^{s'}(\vec{k}') + \cdots \right)$$

$$\Omega_2^{s}(n_k) = \sum_{s_2} \int \left(-3i L^{s s s_2 -s_2}_{k k k_2 -k_2} - 4 \sum_{s_1} \int L^{s s_1 s_2}_{k k_1 k_2} L^{s_1 s -s_2}_{k_1 k -k_2} \right) \left(P \frac{1}{s_1\omega_1+s_2\omega_2-s\omega} + i\pi \operatorname{sgn} t\, \delta(s_1\omega_1+s_2\omega_2-s\omega) \right) \delta(\vec{k}_1+\vec{k}_2-\vec{k})\, d\vec{k}_1 d\vec{k}_2$$

Effectivily : $\omega_k^s \longrightarrow \omega_k^s + \varepsilon^2 \Omega_2^s(\vec{k}) + \cdots$

A frequency modulation

assume it is s independent)

$$4\pi\mathrm{sgn}(t)\sum_{s_1 s_2}\int L^{ss_1s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2}\left(\frac{L^{-s-s_1-s_2}_{-\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}}{n_{\mathbf{k}}}+\frac{L^{s_1s-s_2}_{\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}}{n_{\mathbf{k}_1}}+\frac{L^{s_2s-s_1}_{\mathbf{k}_2\mathbf{k}-\mathbf{k}_1}}{n_{\mathbf{k}_2}}\right)$$
$$\times\,\delta(s_1\omega_1+s_2\omega_2-s\omega)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2. \tag{1.6}$$

The secular terms arising in $Q_2^{ss'\cdots}$ for all other cumulants of order r (that includes order 2 when $s' \neq -s$) take a remarkably simple form,

$$itq_0^{ss'\cdots}(\mathbf{k},\mathbf{k}',\ldots)(\Omega_2^s[n_{\mathbf{k}}]+\Omega_2^{s'}[n_{\mathbf{k}'}]+\cdots), \tag{1.7}$$

where

$$\Omega_2^s[n_{\mathbf{k}}]=\sum_{s_2}\int\left(-3iL^{sss_2-s_2}_{\mathbf{k}\mathbf{k}\mathbf{k}_2-\mathbf{k}_2}-4\sum_{s_1}\int L^{ss_1s_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}L^{s_1s-s_2}_{\mathbf{k}_1\mathbf{k}-\mathbf{k}_2}\right)$$
$$\times\left(\hat{P}\frac{1}{s_1\omega_1+s_2\omega_2-s\omega}+i\pi\mathrm{sgn}(t)\delta(s_1\omega_1+s_2\omega_2-s\omega)\right)$$
$$\times\,\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})n_{\mathbf{k}_2}d\mathbf{k}_2. \tag{1.8}$$

The symbol $\hat{P}$, not to be confused with the permutation symbol $P^{00'}$, represents the Cauchy principal value. To be meaningful, we have to assume that the quantity $h = s_1\omega_1 + s_2\omega_2 - s\omega$ has only simple zeros. If it has multiple zeros, as it does for acoustic waves, the asymptotics are a bit more complicated (Newell and Aucoin, 1971). In writing (1.8), we have reincluded the terms arising from the cubic interaction with coefficient $L^{ss_1s_2s_3}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$. We also note that the last two terms in (1.6) can be written as $2n_{\mathbf{k}}\mathrm{Im}\Omega_2^s$.

The connection with the random phase approximation: Now, we are in a position to discuss the connection with the so-called random phase approximation which certain authors employ in the derivation of the kinetic equation. In calculating $Q^{-ss}(\mathbf{k})$, we found that the secular terms in the various iterates depend only on $n_{\mathbf{k}}$. All terms involving q_0^N, $N \geq 3$, were bounded. Although we did not, we could have ignored them for this part of the calculations. They have no long time cumulative effect. This means that, had we initially expanded the Fourier amplitudes $A^s_{\mathbf{k}}$ as $A^s_{\mathbf{k}0} + \epsilon A^s_{\mathbf{k}1} + \cdots$, then all product averages $\langle A^s_{\mathbf{k}0}A^{s'}_{\mathbf{k}'0}A^{s''}_{\mathbf{k}''0}\ldots\rangle$ could be decomposed as if the zeroth-order amplitudes had random phases or as if they were joint Gaussian so that only products of two point functions survive (Wick's theorem). But, let us emphasize this point: Only the zeroth-order products of the amplitudes can be expanded as if they had random phases. We stress

Asymptotics and resonant manifolds

We frequently have to evaluate

$$\lim_{t\to\infty} \int f(\vec{k},\vec{k}_1,\vec{k}_2)\,\frac{e^{i\hbar(\vec{k},\vec{k}_1,\vec{k}_2)t}-1}{i\hbar(\vec{k},\vec{k}_1,\vec{k}_2)}\,\delta(\vec{k}_1+\vec{k}_2-\vec{k})\,d\vec{k}_1 d\vec{k}_2$$

When $\hbar \equiv s_1\omega(\vec{k}_1)+s_2\omega(\vec{k}_2)-s\omega(\vec{k})$, $s=\pm 1$

Questions:

How does $\hbar$ behave near its zeros?

When $\hbar(\vec{k},\vec{k}_1,\vec{k}_2)=0$, is $\nabla_k \hbar \neq 0$ SIMPLE

Then $\dfrac{e^{i\hbar t}-1}{i\hbar} \sim \pi\,\mathrm{sgn}\,t\,\delta(\hbar)+iP\left(\dfrac{1}{\hbar}\right)$

Case for gravity / surface tension waves, optical waves of diffraction, plasma waves

but for acoustic / shallow water waves $\omega=c|\vec{k}|$ and then $\hbar=0$ has not got simple zeros. More sophisticated asymptotics required.

Asymptotics: Pg 2.

A related question:

The resonant wave manifolds are the sets of $\vec{k}, \vec{k}_1, \vec{k}_2$

where $s_1\omega(k_1) + s_2\omega(\vec{k}_2) - s\omega(\vec{k}) = 0$, $\vec{k}_1 + \vec{k}_2 - \vec{k} = 0$

- Do they "entangle" all $\vec{k}$ space

  e.g. gravity / surface tension / plasma / optical

or

- Do the resonant manifolds foliate $\vec{k}$ space

  e.g. acoustic waves $\omega = c|\vec{k}|$

  where the resonant manifolds are rays in $\vec{k}$ space.

The first closure tells us how "energy" spreads along resonant manifold --- *but* for acoustic waves, a second closure is needed to tell how energy is shared between neighboring rays.

" Were the dinosaurs *frozen* or *fried*?

PREMISES

Premise 3 (P3): Third, we must ensure that various asymptotic expansions for the slow evolution of such two point functions as the waveaction density $n_{\mathbf{k}}$ remain uniformly valid in wavenumber. In its simplest form this means that the ratio of linear (t_L) to nonlinear (t_{NL}) time scales is small at all wavenumbers. It also means that all asymptotic expansions for the slow evolution of the waveaction density $dn_{\mathbf{k}}/dt$, the frequency renormalization which accounts for the slow time behavior of the leading order, higher order cumulants, and for the structure functions remain uniformly valid in wavenumber on almost all relevant solutions. The reason that we require $\frac{t_L}{t_{NL}}(k) \ll 1$ (with $k = |k|$) is that when we look for the long time behavior of integrals such as $\int f(x)\frac{\sin xt}{x}dx$ we want to know that the multiplying function $f(x)$ which will contain products of the waveaction densities $n_{\mathbf{k}}$ is not only smooth in x (wavenumber) but also that it varies slowly in time.

Ocean gravity waves

Hasselmann (1962)

$$T_4[n_k] = \int |L_{kk_1k_2k_3}|^2 n_k n_{k_1} n_{k_2} n_{k_3} (\frac{1}{n_k} + \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}})$$

$$\delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) dk_1 dk_2 dk_3$$

For L localized near $\omega = \omega_1 = \omega_2 = \omega_3$, one can approximate

$$\frac{\partial n_k}{\partial t} = T_4 [n_k]$$

by

$$\frac{\partial N_\omega}{\partial t} = S_0 \frac{\partial^2}{\partial \omega^2} \cdot \omega^{3x_0+2} n^4 \frac{\partial^2}{\partial \omega^2} \frac{1}{n}$$

$$\int N_\omega d\omega = \int n_k d\vec{k}$$

Stationary solutions

$$S_0 \omega^{3x_0+2} n^4 \frac{\partial^2}{\partial \omega^2} \frac{1}{n} = Q\omega + P$$

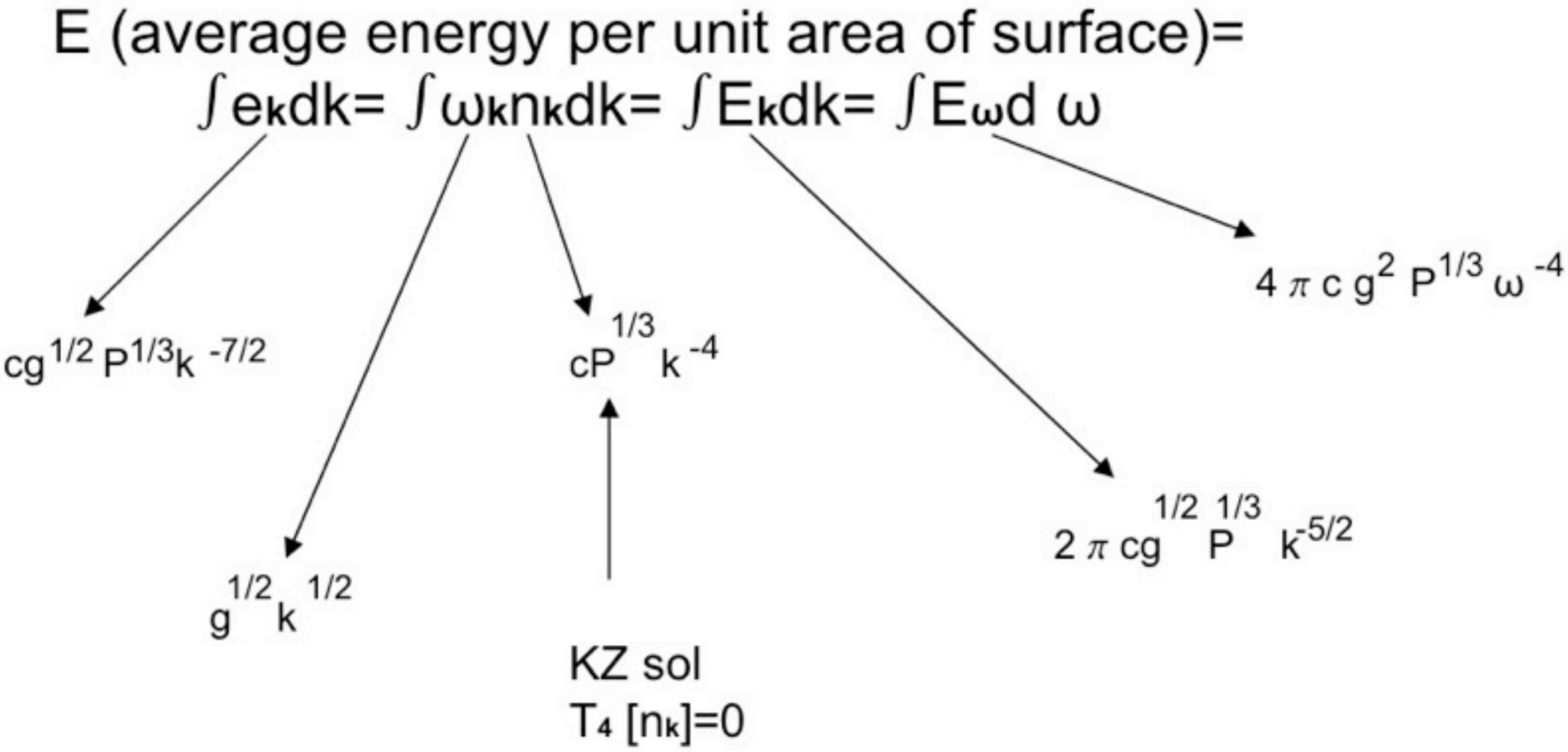
Q, P waveaction & energy fluxes.

If $P = Q = 0$ (isolated system)

$$n = \frac{T}{\omega + \mu}$$

the thermodynamic solutions.

Ocean gravity waves



E (average energy per unit area of surface)=

$$\int e_k dk = \int \omega_k n_k dk = \int E_k dk = \int E_\omega d\,\omega$$

$cg^{1/2}P^{1/3}k^{-7/2}$

$g^{1/2}k^{1/2}$

$cP^{1/3}k^{-4}$

KZ sol

$T_4\,[n_k]=0$

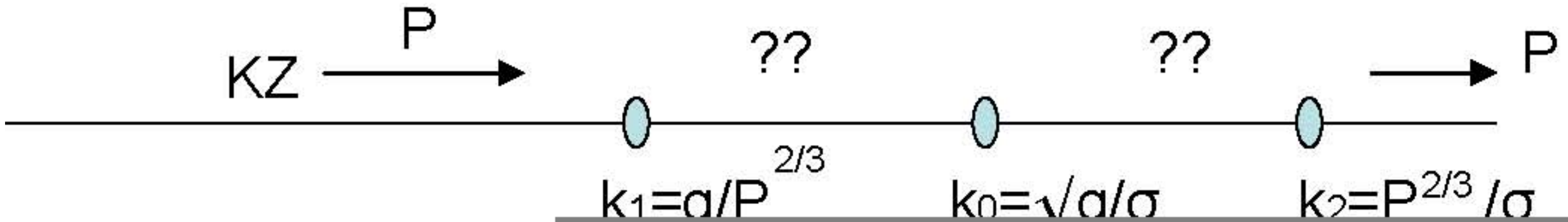
$2\pi c g^{1/2}P^{1/3}k^{-5/2}$

$4\pi c g^{2}P^{1/3}\omega^{-4}$

Breakdown!

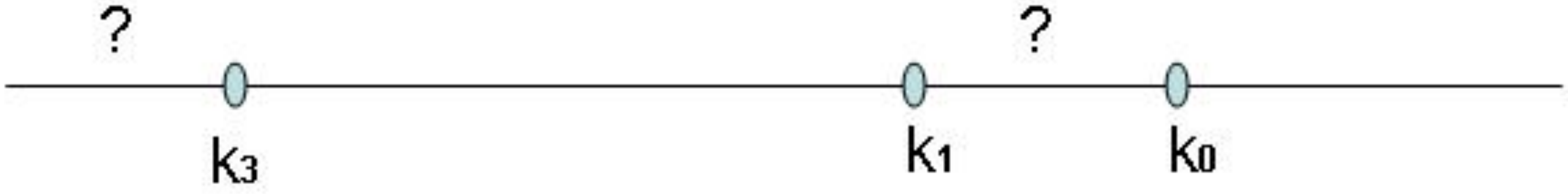
The markers of a successful closure are

- $\frac{t_L}{t_{NL}} = \frac{1}{\omega_k}\frac{1}{n_k}\frac{dn_k}{dt} \simeq c^2 P^{2/3}\frac{k}{g} \ll 1$
- $\frac{T_6[n_k]}{T_4[n_k]}, \frac{\Omega_k[n_k]}{w_k}, \ldots, \simeq cP^{1/3}\frac{k^{1/2}}{g^{1/2}} \ll 1$
- $\frac{S_4-3S_2^2}{S_2^2}, \ldots, \simeq cP^{1/3}\frac{1}{g^{1/2}r^{1/2}} \ll 1$



Breakdown!

The finite flux solutions of wave turbulence are almost never uniformly valid for all wavenumbers. They almost always fail at very high or very low wavenumbers.



Optical waves ← of diffraction and formation of condensates/collapses

Ocean waves and whitecapping

What new solutions obtain in breakdown regions?

The generalized Phillips' spectrum (GPS)

We (ACN VEZ, Phys. Lett. A 372, 4230-4233 (2008)) argue that the GPS can play a central role. It has four important properties.

- It is the only spectrum for which symmetries of the original governing equations are inherited by the asymptotic statistics. This in fact will serve as the definition of the GPS.
- It is the unique spectrum on which wave turbulence is uniformly valid at all wavenumbers.

The generalized Phillips' spectrum (GPS)

- It is a solution of $T_4[n_k] - \gamma_k[n_k] = 0$, i.e. a balance of nonlinear transfer and dissipation and independent of energy flux.(i.e. can absorb any excess flux)
- It is connected with whitecapping and Phillips' picture.

Inheriting symmetries

$$\frac{dA_k^s}{dt} - is\underset{\substack{|\\ \alpha}}{\omega_k} A_k^s = \sum_{r=2} \int \underset{\substack{|\\ \gamma_r}}{L}{}_{k\ldots k_r}^{s\ldots s_r} A_{k1}^{s1} \ldots A_{k_r}^{s_r} \delta(k_1+\ldots+k_r-k)\,dk_1\ldots dk_r$$

degrees of homogeneity

$$\mathbf{k} \rightarrow \mathbf{K} = \lambda \mathbf{k}, \qquad t \rightarrow T = \lambda^{-\alpha} t, \qquad A_k^s \rightarrow \lambda^b B_K^s$$

$$\lambda^{b-\alpha} \qquad\qquad \lambda^{-\gamma_r} \quad \lambda^{rb} \qquad\qquad \lambda^{-(r-1)d}$$

$$\Longrightarrow b = d + \frac{\gamma_r - \alpha}{r-1} \quad , \text{ all } r$$

$$< A_k^s \; A_{k'}^{-s} > \; = \delta(\mathbf{k} + \mathbf{k}')\, k^{-\alpha x} \xleftarrow{\quad} n_k$$

$$\Longrightarrow \lambda^{2b} < B_K^s \, B_{K'}^{-s} > \; = \lambda^d \delta(\mathbf{K} + \mathbf{K}')K^{-\alpha x} \lambda^{-\alpha x}$$

$$\Longrightarrow \alpha x = 2b\text{-}d = d + 2\frac{\gamma_r - \alpha}{r-1} \qquad \text{GPS}$$

Inheriting symmetries

Gravity waves:

$\alpha = \frac{1}{2}, \gamma_2 = \frac{7}{4}, \gamma_3 = 3, d = 2 \Rightarrow n_k \sim k^{-9/2}, < \eta_k^2 > \sim k^{-4}.$

Capillary waves:

$\alpha = \frac{3}{2}, \gamma_2 = \frac{9}{4}, \gamma_3 = 3, d = 2 \Rightarrow n_k \sim k^{-7/2}, < \eta_k^2 > \sim k^{-4}.$

WT U valid all k: $n_k \sim k^{-\alpha x}$, $\alpha x = ?$

$$\frac{dn_k}{dt} = T_2[n_k] + T_4[n_k] + T_6[n_k] + \cdots$$

$$\boldsymbol{s}\omega_k \longrightarrow \boldsymbol{s}\omega_k + \Omega^s_k[n_k] + \cdots \qquad U \text{ valid in } k?$$

$$\frac{t_L}{t_{NL}} = \frac{1}{\omega_k}\frac{1}{n_k}\frac{dn_k}{dt} = \frac{\boldsymbol{s}^{\cdot}_k}{\omega_k} \ll 1 \text{ all } k\ ?$$

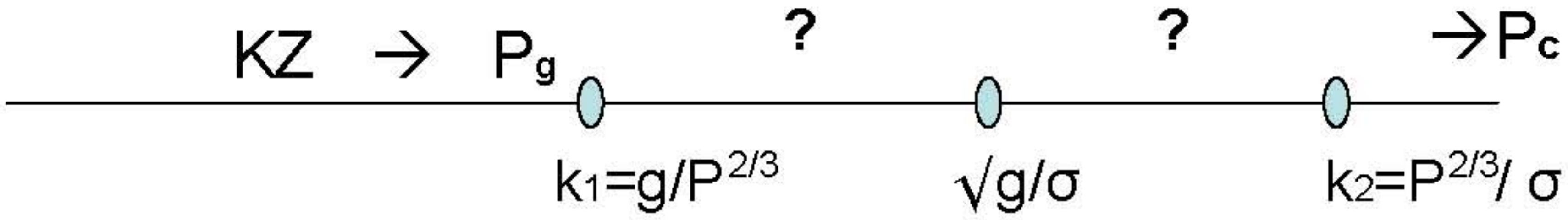
$$\frac{S_4 - 3S_2^2}{S_2^2} \ll 1 \text{ all } r?$$

$$\frac{t_L}{t_{NL}} \qquad n_k = ck^{-\alpha x} \qquad \frac{T_4}{\omega_k n_k} \sim \frac{k^{2\gamma r} c^3 k^{-3\alpha x} k^{-\alpha} k^{2d}}{k^{\alpha} ck^{-\alpha x}} \sim c^2 k^{2(\gamma_3 + d - \alpha - \alpha x)}$$

WT U valid all k: $n_k \sim k^{-\alpha x}$, $\alpha x = ?$

On GPS, $\alpha x = \gamma_3 + d - \alpha$, $\frac{t_L}{t_{NL}}$ k independent

On KZ, $\alpha x = \frac{2\gamma_3}{3} + d$, $\frac{t_L}{t_{NL}} \sim cP^{2/3}k^{2(\frac{\gamma_3}{3}-\alpha)} \sim P^{2/3}\frac{k}{g}$



KZ → P_g

?

?

→P_c

$k_1=g/P^{2/3}$

$\sqrt{g}/\sigma$

$k_2=P^{2/3}/\ \sigma$

GPS solves $T[n_k] - \gamma[n_k] = 0$

From $\frac{dn_k}{dt} = T[n_k] - \gamma[n_k]$, angle average to obtain

$$\frac{dE_k}{dt} = -\frac{\partial p(k)}{\partial k} - \gamma_k$$

where

$$\int \omega_k n_k dk = \int E_k dk, \quad -\frac{\partial p(k)}{\partial k} = < \omega_k T[n_k] >$$

Let RHS=0 in integral sense

$$\int_{k_1}^{\infty} -\frac{\partial p(k)}{\partial k} dk = p(k_1) = \int_{k_1}^{\infty} \gamma_k dk = P$$

breakdown wavenumber

all dissipation
between k_1 and ∞
absorbs flux P

GPS solves $T[n_k] - \gamma[n_k] = 0$

But on $n_k = Ck^{-\alpha x}, p(k_1) = C^3\mathbf{I}(x)k_1^{3\alpha(x_{KZ}-x)}$

$$C^3\mathbf{I}(x)k_1^{3\alpha(x_{KZ}-x)} = P = k_1^{-\gamma_3+3\alpha}$$

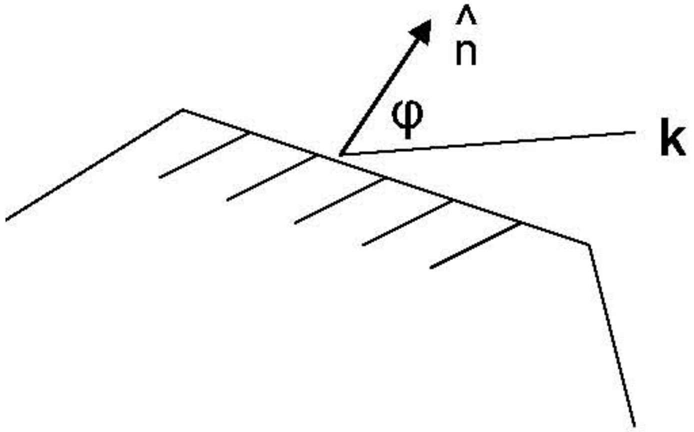
$$\Rightarrow \alpha x = \gamma_3 + d - \alpha, \, GPS.$$

Connection with whitecaps

Consider sea surface dominated by a series of wedge shapes of finite length.

$$\eta(x,y) \sim \frac{1}{2s} e^{-s|x|} e^{-\frac{y^2}{L^2}}$$

$$< \eta_k^2 > \propto \frac{1}{(s^2 + k^2 \cos^2 \varphi)^2} e^{-\frac{k^2 L^2 \sin^2 \varphi}{2}}$$



Connection with whitecaps

For s fixed, average over $\varphi \Rightarrow k^{-5}$, not Phillips'!!!
But averaging over s, $s > k_1, \ldots,$

$$\frac{L}{k^3}\int_0^{\frac{\pi}{2}} e^{-\frac{k^2L^2\sin^2\varphi}{2}} \frac{1}{\cos^3\varphi}[\frac{\pi}{4}-\frac{1}{2}\tan^{-1}\frac{k_1}{k\cos\varphi}-\frac{1}{2}\frac{kk_1\cos\varphi}{k_1^2+k^2\cos^2\varphi}]d$$

for $kL \gg 1, < \eta_k^2 > \sim k^{-4}$, Phillips'!!!

MMT equation

$$i\frac{\partial u}{\partial t} = Lu + \lambda u^2 u^* \ , \ Le^{ikx} = \omega_k e^{ikx}, \omega_k = \sqrt{g|k|}.$$

$$\begin{aligned} T_4[n_{\mathbf{k}}] = 4\pi\lambda^2 \ \int \ & n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} \\ \times \ & \left(\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} - \frac{1}{n_{\mathbf{k}_3}} \right) \\ \times \ & \delta(\omega + \omega_1 - \omega_2 - \omega_3) \\ \times \ & \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123} \end{aligned}$$

$$\tilde{\omega}_k = \omega_k + 2\lambda \int n_{\mathbf{k}} d\mathbf{k}$$

Possible answers to Q2.

These lessons Q2a,b,c,d beg for an answer to question 2. How might we have known, beforehand, whether the long time dynamics of an initially weakly nonlinear system is dominated by wavetrains, or coherent objects, or a mixture of both? We have suggested one possible condition as premise P4; namely that the governing equation (1.1) is asymptotically linear if $A^s_{\mathbf{k}}$ is an ordinary function. And now we suggest another condition based on a new and truly remarkable result. We have found that, in certain rare circumstances, the KZ solution can be unstable to disturbances which spontaneously break the symmetry of spatial homogeneity (preprint by the authors with V.E. Zakharov), namely break P1!

What we have found is that the KZ solution $n_k \sim k^{-1}$ for MMT is unstable when $\lambda = +1$ and neutrally stable when $\lambda = -1$ when perturbations which assume that the two point correlations such as $\langle u^s(\mathbf{x})u^{-s}(\mathbf{x}+\mathbf{r})\rangle$ are allowed to depend weakly on the base coordinate $\mathbf{x}$. The analysis uses the Vlasov form of the kinetic equation described at the end of section 2. The reader might ask how can it be that for one sign of λ the KZ solution is stable and for the other unstable since the kinetic equations for both cases are the same, depending only on λ^2. But whereas the kinetic equations are the same, the first nonlinear frequency corrections depend on the sign of λ and in the nonlinear dependence of ω_k on $n_{\mathbf{k}}$ which determines stability or instability. In addition we find that even when $\lambda = 1$ but the dimension of the system is two or greater, then again the KZ spectrum is neutrally stable against perturbations which break spatial homogeneity.

In a numerical experiment, we created the k^{-1} spectrum by allowing an ensemble of solutions of (1.27) reach a statistically steady state for the choice of sign $\lambda = -1$. Then we switched the sign of λ and watched as coherent structures resembling ultra short pulse waves ("rogue waves") came to dominate the dynamics and the statistics. The instability is slow in the sense that it does not occur on a time scale $(\epsilon^2\omega_0)^{-1}$, ω_0 the frequency of the spectral peak associated with the frequency renormalization, but rather on the same time scale $(\epsilon^4\omega_0)^{-1}$ over which the four wave resonances act.

So, we introduce premise P5. All KZ solutions are stable against perturbations which spontaneously break the spatial homogeneity symmetry. It would be very interesting to know if, along with P1, P2, P3, either or both of these two conditions P4, P5 give sufficient conditions for wave turbulence closure to obtain and if, and in what way, they may be connected.

In the introduction we made the point that the wave turbulence closure depends on very mild statistical assumptions. Certainly if P5 holds, one can be sure that the validity of P1 is not in doubt. P2, the fact that the initial fields are uncorrelated at widely separated points is also very mild. P3 is a serious precondition but its validity can always be tested. P4, which is not an a priori condition on the statistics but on the dynamics of individual solutions, may be stronger than one requires to have the long time behavior of the random system dominated by resonantly interacting

PREMISES

Premise 4 (P4): This premise says that one must test the deterministic theory first. If the field remains asymptotically linear (which might be tested by numerical simulations), we might surmise that this would rule out the appearance of coherent structures also dominating the long time behavior of the random system. The thinking here is that the deterministic problem would rule out resonances creating secular behavior (because the wavepackets are finite in length and so resonances do not produce long time cumulative effects) but not the appearance of coherent structures. If the latter do not appear in the deterministic system, the argument is that they will play no role in statistical ensembles either.

Premise 5 (P5): All KZ solutions are stable against perturbations which spontaneously break the spatial homogeneity symmetry.

$$n_{\mathbf{k}} = n_0(k) = DP^{1/3}k^{-d}$$

$$\frac{\partial n_{\mathbf{k}}}{\partial t} + \nabla_{\mathbf{k}}\tilde{\omega}_k \cdot \nabla_{\mathrm{x}} n_{\mathbf{k}} - \nabla_{\mathrm{x}}\tilde{\omega}_k \cdot \nabla_{\mathbf{k}} n_{\mathbf{k}} = \sum_{r=1} T_{2r}[n_{\mathbf{k}}]$$

Inserting $n_0(k) + \Delta(\mathbf{k})\exp(i\mathbf{K}\cdot\mathbf{x} - i\Omega t)$,

$$i\Delta(\mathbf{k})(\omega'_k\hat{\mathbf{k}}\cdot\hat{\mathbf{K}} - c)K - 2i\hat{\mathbf{k}}\cdot\hat{\mathbf{K}}Kn'_0(k)\lambda\int\Delta(\mathbf{k})d\mathbf{k} = \delta T_4$$

where $K = |\mathbf{K}|$, $\hat{\mathbf{k}} = \mathbf{k}/k$, $\hat{\mathbf{K}} = \mathbf{K}/K$, $n'_0(k) = dn_0/dk$. $c = \Omega/K$ is the phase velocity of the modulation and $\omega'_k = d\omega_k/dk$ is the group velocity of linear waves. Complex values of c and Ω will signal instability. δT_4 is a linear functional of $\Delta(\mathbf{k})$. To begin we ignore δT_4, and we will discuss its small influence later. Integrating over $\mathbf{k}$ we obtain

$$1 = 2\lambda\int\frac{n'_0(k)\hat{\mathbf{k}}\cdot\hat{\mathbf{K}}}{\omega'_k\hat{\mathbf{k}}\cdot\hat{\mathbf{K}} - c}d\mathbf{k}$$

$$1 = 2\lambda \int_{-\infty}^{\infty} \frac{n_0(k)\omega_k''}{(\omega_k' - c)^2} dk.$$

$$1 + \alpha \int_{1}^{\infty} \frac{z^2}{\sigma^2} \left(\frac{1}{(z-\sigma)^2} + \frac{1}{(z+\sigma)^2} \right) d\sigma = 0.$$

$$z = \frac{g}{2c\omega_0} = \omega_0'/c\,,\ \alpha = 4\lambda D\, P^{1/3}/\omega_0$$

$$1 + \alpha \left(2 + \frac{2}{z} \ln \frac{1-z}{1+z} + \frac{1}{1-z} + \frac{1}{1+z} \right) = 0.$$

In 2D

$$\frac{1}{8\pi\alpha} = \frac{1}{z} \sin^{-1} z - \frac{1}{4} \frac{1}{\sqrt{1-z^2}} - \frac{3}{4}$$

for $z = \omega_0'/c$, or with $z = \sin(\zeta)$,

$$\frac{1}{8\pi\alpha} = \zeta \mathrm{cosec}(\zeta) - \frac{1}{4} \sec(\zeta) - \frac{3}{4}.$$

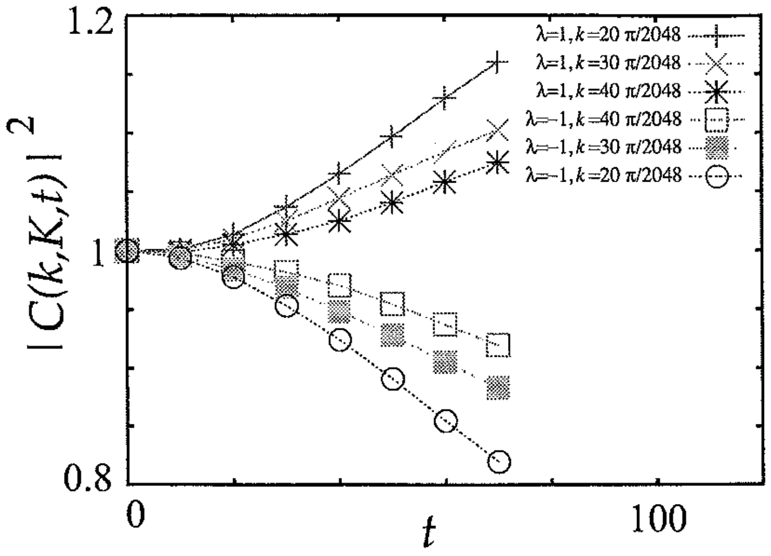


FIG. 2: Time evolution of the correlation $|C(k,K,t)|^2 = |\langle A_k(t)A^*_{k+K}(t)\rangle|^2/|\langle A_k(0)A^*_{k+K}(0)\rangle|^2$ with $K = 6\pi/2048$ for an ensemble of 400,000 trajectories for the equation (2) with $\lambda = 1$ and with $\lambda = -1$. The initial conditions are Kolmogorov-Zakharov distributed $n_k \sim k^{-1}$ with a Gaussian amplitude distribution and random phases for $|k| \geq 20\pi/2048$ and $n_k = 0$ for $|k| < 20\pi/2048$ (the wavenumber space is $-\pi < k \leq \pi$). A small spatial modulation is superimposed on these random initial conditions, and this modulation is the same for each member of the ensemble. The system is not externally damped or driven. This correlation grows for $\lambda = 1$, reflecting an instability of wave turbulence against spatially inhomogeneous perturbations. There is no such instability for $\lambda = -1$, and the correlation decays.

Open Challenges

Acoustic Turbulence, Isotropy or Shocks?
The resonant manifolds for the dispersion relation $\omega = c|k|$ are rays in wavevector space. The first closure transfers spectral energy along but not between the rays. Given an initial anisotropic energy distribution, do the nonlinear interactions of the next closure lead to an isotropic distribution or to condensation along particular rays which would likely produce fully nonlinear shocks (L'vov et al. (1997))? Or, to use a more colorful vernacular: Were the dinosaurs frozen or fried?

Energy Exchange Times.
For a discrete set of interacting triads, the nonlinear energy exchange time is ϵ^{-1}. For a continuum set of such triads, "cancellations" cause this time to be extended to ϵ^{-2}. Why?

Condensate Formation,
modeled by the defocussing ($\lambda = -1$) NLS equation, is an open and hot topic.

Wave. Turbulence in Astrophysics.
Magnetized plasmas, found in the solar corona, solar wind and earth's magneto-sphere support waves and, like ocean waves, have a continuum of scales (up to 18 decades!) and are a natural playground for wave turbulence.

Continuum Limit of Finite Dimensional Wave Turbulence.

Open Challenges

A Priori Conditions for Wave Turbulence.
Can one find mathematically rigorous a priori conditions on the governing equation (1.1) or its statistical hierarchy which guarantees that wave turbulence theory will obtain?

Homogeneity.
Is broken spatial homogeneity (PI) a potential problem for all turbulence theories?

Anomalous Exponents.
Are all finite capacity Kolmogorov solutions reached with anomalous exponents? Do they have anything to do with positive entropy production?