Evidence of resonant interactions in ocean waves

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Outline

- Motivation
- An introduction to resonant interactions in water waves
- Deterministic four-wave resonances
- Four-wave resonances in random waves
- Experiments from Acqua Alta Tower in the Adriatic Sea
- Verification of some of the predictions of WT theory from data
- Discussion

Owen M. Phillips (1930-2010)

On the dynamics of unsteady gravity waves of finite amplitude

Part 1. The elementary interactions

By O. M. PHILLIPS
Mechanics Department, The Johns Hopkins University,
Baltimore, Maryland

(Received 12 March 1960)

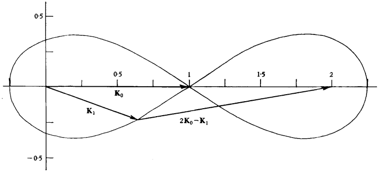
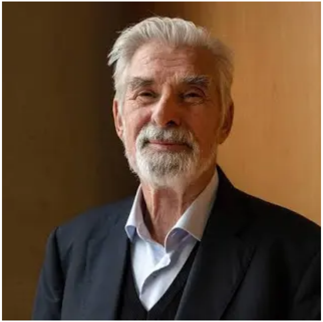


FIGURE 4. The resonance loop for third-order binary interactions. The wave-number $\mathbf{K}_1$ interacts with the bound secondary component associated with $\mathbf{K}_0$ to produce a developing component of wave-number $2\mathbf{K}_0 - \mathbf{K}_1$. The arrows represent the directions of propagation.

$$2\mathbf{k_0} = \mathbf{k_1} + \mathbf{k_2}$$

$$2\omega(k_0) = \omega(k_1) + \omega(k_2)$$

K. Hasselmann (1931-) and V. E. Zakharov (1939-2023)



The Wave Kinetic equation

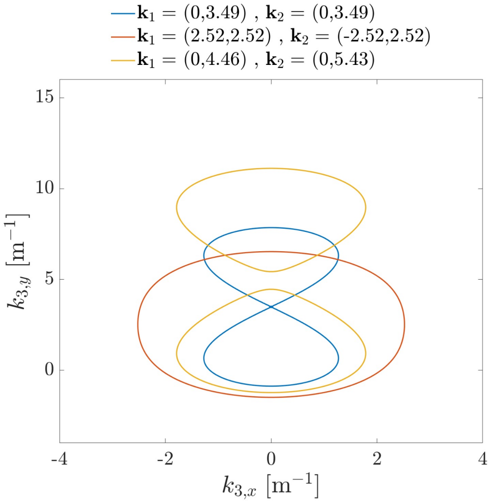
$$\frac{\partial n(\mathbf{k}_1,t)}{\partial t} = \epsilon^4 \int_{-\infty}^{\infty} |T_{1234}|^2 n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\Delta k) \delta(\Delta \omega) d\mathbf{k}_{2,3,4}$$

where

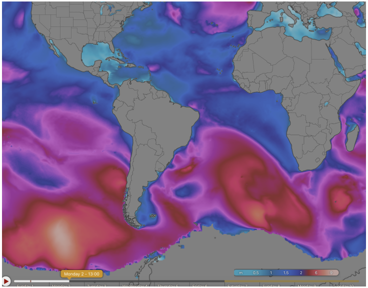
$$\delta(\Delta k) = \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$

$$\delta(\Delta \omega) = \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

Examples of resonant manifolds



The motivation: ocean wave forecasting



$$\frac{\partial n(\mathbf{k}_1, \mathbf{x}, t)}{\partial t} + v_{\mathbf{k_1}} \cdot \frac{\partial n(\mathbf{k}_1, \mathbf{x}, t)}{\partial \mathbf{x}} = S_{nl} + S_{diss} + S_{in}$$

where S_{nl} is the collision integral:

$$S_{nl}(\mathbf{k}_1, \mathbf{x}, t) = \int_{-\infty}^{\infty} |T_{1234}|^2 n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\Delta k) \delta(\Delta \omega) d\mathbf{k}_{2,3,4}$$

$$\delta(\Delta k) = \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$
$$\delta(\Delta \omega) = \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

The deterministic dynamics: the Euler equations

- The fundamental assumption is that the flow is ideal: inviscid and irrotational
- The dynamics of the atmosphere is decouple from the ocean
- The theory is expressed in terms of a set of PDEs for the surface elevation, $\eta = \eta(\mathbf{x}, t)$, and the velocity potential, $\phi(\mathbf{x}, z, t)$
- The equations of motion reduce to the Laplace equation $+$ boundary conditions
- The problem is fully non linear and not amable for analytic treatment

The weakly nonlinear expansion and the "normal variable" in Fourier space

- To proceed, it is necessary to Taylor expand the solution for the velocity potential around the flat surface (weakly nonlinear assumption)
- Such procedure allows to express the potential as a function of the surface and the potential at the surface: the new variables are $\eta = \eta(\mathbf{x}, t)$ and $\psi = \psi(\mathbf{x}, t) = \phi(\mathbf{x}, z = \eta, t)$
- One then assumes a periodic box $L \times L$ and use Fourier series to express the variables in terms of Fourier amplitudes: $\eta_{\mathbf{k}}(t)$ and $\psi_{\mathbf{k}}(t)$
- The following variable is introduced which is related to the wave action

$$a_{\mathbf{k}}(t) = \sqrt{\frac{g}{2\omega_k}}\eta_{\mathbf{k}}(t) + i\sqrt{\frac{2\omega_k}{g}}\psi_{\mathbf{k}}(t)$$

The wave-wave interaction equation

$$i\frac{da_1}{dt} = \omega_1 a_1 + \epsilon\ 3\text{W.I.} + \epsilon^2 4\text{W.I.} + \epsilon^3 .....$$

where

$$3\text{W.I.} = \sum_{k_2,k_3} \left(V^{(1)}_{123} a_2 a_3 \delta^{23}_{1} + 2V^{(1)}_{321} a_2^* a_3 \delta^{3}_{12} + V^{(3)}_{123} a_2^* a_3^* \delta_{123} \right)$$

$$4\text{W.I.} = \sum_{k_2,k_3,k_4} \big(T^{(1)}_{1234} a_2 a_3 a_4 \delta^{234}_{1} + T^{(2)}_{1234} a_2^* a_3 a_4 \delta^{34}_{12} +$$

$$+ 3T^{(1)}_{4321} a_2^* a_3^* a_4 \delta^{4}_{123} + T^{(4)}_{1234} a_2^* a_3^* a_4^* \delta_{1234} \big)$$

where the notation for the Kronecker deltas is the following:

$$\delta^{23}_{1} = \begin{cases} 1 & \text{if} \quad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3 \\ 0 & else \end{cases} \qquad \delta^{34}_{12} = \begin{cases} 1 & \text{if} \quad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \\ 0 & else \end{cases}$$

Removing nonresonant interactions (the normal form expansion)

The following near-identity transformation (from $a_{\mathbf{k}}$ to $b_{\mathbf{k}}$) is introduced to remove nonresonant interactions:

$$a_1 = b_1 + \epsilon \sum_{k_2,k_3} \left(A^{(1)}_{123} b_2 b_3 \delta_1^{23} + A^{(2)}_{123} b_2^* b_3 \delta_{12}^{3} + A^{(3)}_{123} b_2^* b_3^* \delta_{123} + \epsilon^2 .... \right)$$

to obtain the celebrated Zakharov equation (in interaction representation)

$$i \frac{db_1}{dt} = \epsilon^2 \sum_{k_2,k_3,k_4} T_{1234} b_2^* b_3 b_4 \delta_{12}^{34} e^{i \Delta \omega_{12}^{34} t}$$

with $\Delta\omega_{12}^{34} = \omega_1 + \omega_2 - \omega_3 - \omega_4$.

The Zakharov equation is the starting point for deriving:

i) reduced models for resonant interactions

ii) statistical theory of surface gravity waves

Reduced model for 4-wave resonant interactions (Benny, JFM 1962)

We consider three waves

$$b(\mathbf{k}, t) = b_1(t)\delta(\mathbf{k} - \mathbf{k}_1) + b_3(t)\delta(\mathbf{k} - \mathbf{k}_3) + b_4(t)\delta(\mathbf{k} - \mathbf{k}_4)$$

whose wavenumbers satisfy the particular (degenerated) 4-wave resonant interaction:

$$\begin{aligned} 2\mathbf{k_1} &= \mathbf{k_3} + \mathbf{k_4} \\ 2\omega(k_1) &= \omega(k_3) + \omega(k_4) \end{aligned}$$

to obtain:

$$\begin{aligned} i\dot{b}_1 &= \epsilon^2[(T_{1111}|b_1|^2 + 2T_{1313}|b_3|^2 + 2T_{1414}|b_4|^2)b_1 + 2T_{1134}b_1^*b_3b_4] \\ i\dot{b}_3 &= \epsilon^2[(2T_{1313}|b_1|^2 + T_{3333}|b_3|^2 + 2T_{3434}|b_4|^2)b_3 + 2T_{1134}b_1^2b_4^*] \\ i\dot{b}_4 &= \epsilon^2[(2T_{1414}|b_1|^2 + 2T_{3434}|b_3|^2 + 2T_{4444}|b_4|^2)b_4 + 2T_{1134}b_1^2b_3^*] \end{aligned}$$

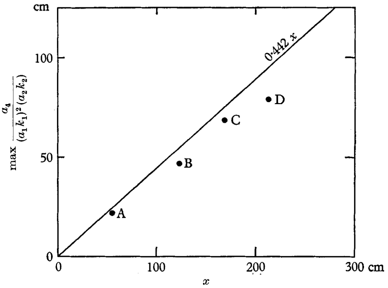
Approximate solution of the reduced model

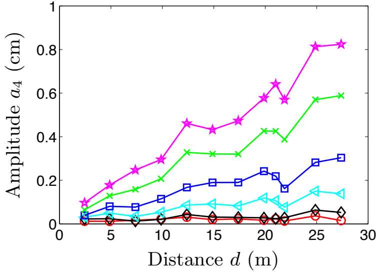
Assuming $b_4(t=0)=0$ and solving asymptotically, the solution for b_4 can be found at order ϵ^2

$$b_4 = -i2T_{1134}b_1^2 b_3 t$$

The "daughter" wave grows linearly in time, for small time.

- Such growth (in space rather than in time) has been verified experimentally starting from the work of M.S. Longuet-Higgins and D. Smith, JFM 1966, and more recently by F. Bonnefoy et al. JFM 2016.





Statistical theory of surface gravity waves: A sketch of the derivation of the WKE

We look for an evolution equation for the wave action, averaging over initial data assumed to be *i.i.d. random variables*:

$$\frac{d\langle |b_1|^2\rangle}{dt} = \epsilon^2 2\mathfrak{Im}\left[\sum_{2,3,4} T_{1234}\langle b_1^* b_2^* b_3 b_4\rangle e^{i\Delta\omega_{12}^{34}t}\delta_{12}^{34}\right]$$

$$\frac{d\langle b_1^* b_2^* b_3 b_4\rangle}{dt} = \epsilon^2 F(\langle b_1^* b_2 b_3 b_4 b_5^* b_6^*\rangle, \langle b_1^* b_2^* b_3 b_4^* b_5 b_6\rangle; t)$$

$$\frac{d\langle b_1^* b_2 b_3 b_4 b_5^* b_6^*\rangle}{dt} = O(\epsilon^2)$$

The end result (after taking the large box limit and sending $\epsilon \to 0$) is that

$$2\mathfrak{Im}\left[\sum_{2,3,4} T_{1234}\langle b_1^* b_2^* b_3 b_4\rangle e^{i\Delta\omega_{12}^{34}t}\delta_{12}^{34}\right] \to \epsilon^2 S_{nl}$$

The fourth-order correlator should be different from zero only on the resonant manifold

The Kolmogorov-Zakharov solution of the WKE: direct energy cascade

$$\frac{\partial n(\mathbf{k}_1,t)}{\partial t} = \epsilon^4 \int_{-\infty}^{\infty} |T_{1234}|^2 n_1 n_2 n_3 n_4 \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right) \delta(\Delta k)\delta(\Delta\omega) d\mathbf{k}_{2,3,4}$$

Constant flux solution

- Direct cascade of energy

$$n(\mathbf{k}) = CP^{1/3}k^{-4}$$

  which, in terms of one dimensional energy spectral density function, corresponds to

$$E(k) \sim k^{-2.5}$$

  or, using the dispersion relation, in frequency

$$E(\omega) \sim \omega^{-4}$$

- The solution is isotropic, and it exists only in the presence of forcing and dissipation

Numerical verification of KZ solution

O. M., Osborne, A.R., Serio, M., Resio, D., Pushkarev, A., Zakharov, V.E. and Brandini, C. Freely decaying weak turbulence for sea surface gravity waves. Physical Review Letters, 2002

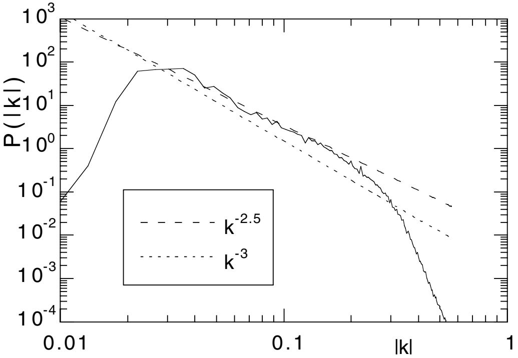


FIG. 2. Wave spectrum at $t = 4$ h. A $k^{-2.5}$ (dashed line) and a k^{-3} (dotted line) power law are also plotted.

Numerical verification of KZ solution

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Weak Turbulent Kolmogorov Spectrum for Surface Gravity Waves

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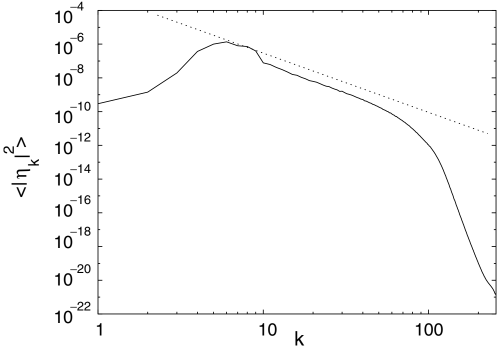

FIG. 2. Averaged spectrum of surface elevation $\langle|\eta_k|^2\rangle$. Line $\sim k^{7/2}$ is also shown.

The ω^{-4} and ω^{-5} controversy

- O. Phillips in 1957 (JFM), based on a dimensional argument, proposed a frequency decay of the spectrum of the type:

$$E(\omega) \sim g^2 \omega^{-5}$$

- K. Hasselmann in 1973 performed experiments in the North Sea and parametrized the spectrum with a ω^{-5} tail
- These results were in disagreement with Zakharov theory

Experimental evidence of the ω^{-4} spectrum in the ocean

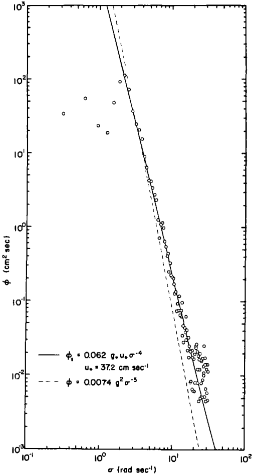

Fig. 12. An example of power spectrum for Run no. 18 of Series A.

Taken from Kawai, S., Okada, K. and Toba, Y. Journal of the Oceanographical Society of Japan (1977)

Experimental results in the ocean: the ω^{-4} spectrum

JOURNAL OF PHYSICAL OCEANOGRAPHY VOLUME 13

On the Theory of the Equilibrium Range in the Spectrum of Wind-Generated Gravity Waves

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Department of Earth and Planetary Sciences, The Johns Hopkins University, Baltimore, MD 21218

(Manuscript received 23 November 1982, in final form 14 February 1983)

ABSTRACT

It is shown that an exact analog of Kolmogoroff's spectrum in a random field of weakly nonlinear surface gravity waves gives a spectral form for frequency spectra $S(\omega) \sim \omega^{-4}$ in close agreement with the results of recent observational studies. The proposed theory also indicates the existence of a "transitional" range of wavenumbers (frequencies) where the deviation from Kolmogoroff's equilibrium is due to gravitational instability (wave breaking). Because of this it is suggested that the equilibrium form for the spectrum of wind-generated waves has two asymptotic regimes: Kolmogoroff's *and* Phillips' type of equilibrium with a relatively rapid transition from the first to the second. The experimental data favor such an interpretation.

Experimental results in the ocean: the ω^{-4} spectrum

JOURNAL OF PHYSICAL OCEANOGRAPHY VOLUME 17

A Reanalysis of the Spectra Observed in JONSWAP

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(Manuscript received 28 July 1986, in final form 4 March 1987)

ABSTRACT

The frequency spectra of wind-driven waves observed during JONSWAP are reanalyzed to establish whether the Toba formulation for the high-frequency tail ($\sim f^{-4}$) fits the data better than the Phillips formulation ($\sim f^{-5}$) used originally in the JONSWAP project. The results indicate that the f^{-4} tail provides a statistically better fit to the observed spectra. The proportionality factor in Toba's spectrum, which is theoretically expected to be a universal constant, is found to be uncorrelated with the growth stage of the waves. There is a relatively large scatter in the observed values, which can partly be ascribed to the influence of tidal currents.

Lab experiments related to direct energy cascade spectrum

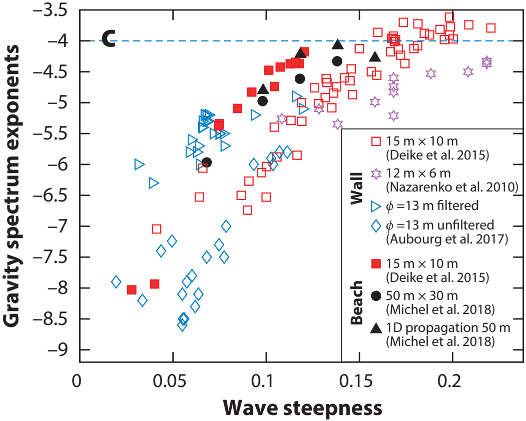


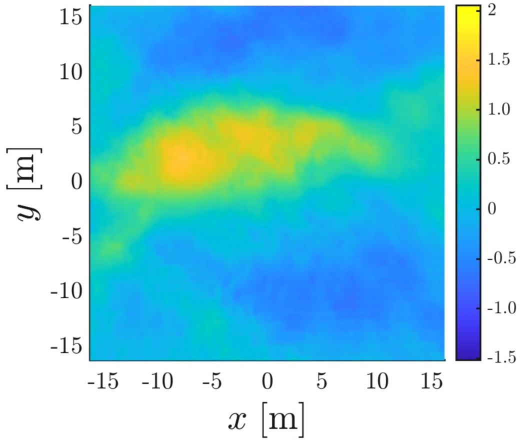
Figure from: Falcon, E. and Mordant, N., Experiments in surface gravity-capillary wave turbulence. Annual Review of Fluid Mechanics (2022)

The data from Acqua Alta Oceanographic Tower

Using two cameras mounted at 12.5 m above the mean sea surface pointing to the same area, it is possible to measure the surface elevation in space and time

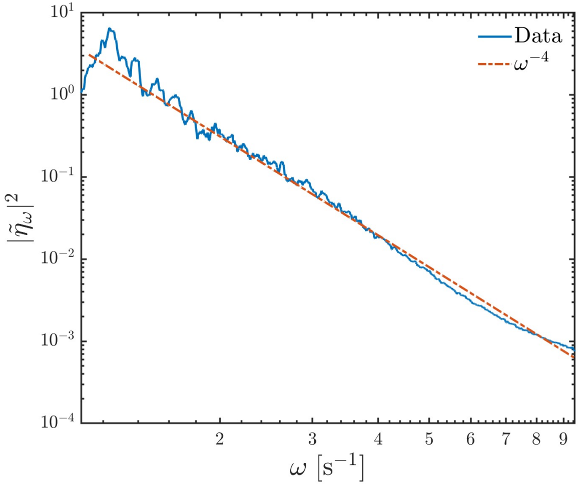
- The data set are described in Guimarães, P. V., et al. "A data set of sea surface stereo images to resolve space-time wave fields. Sci. Data, 7, 145." (2020).

The data from Acqua Alta Oceanographic Tower

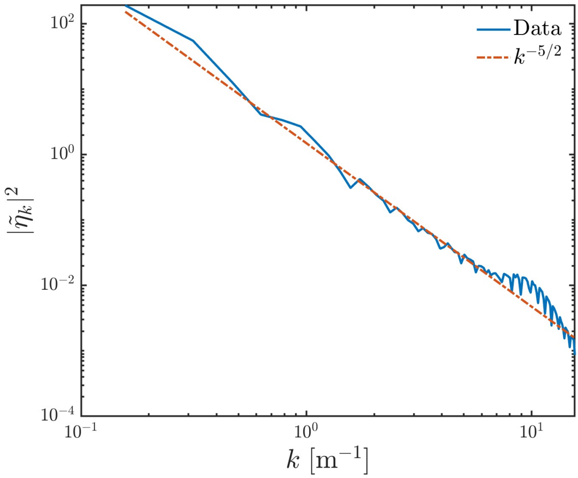


Spatial resolution: 0.2 m
Temporal resolution: 1/12 s
Total temporal length: 20000 frames (almost 28 minutes)

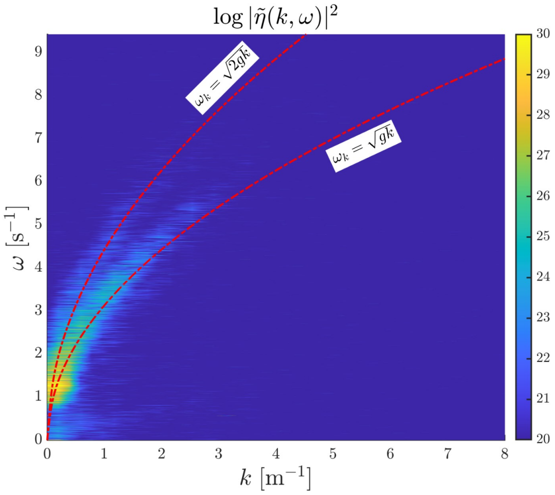
The frequency spectrum



Wave number spectrum

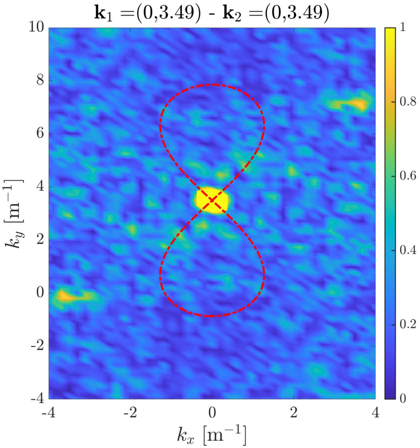


The $(k-\omega)$ spectrum

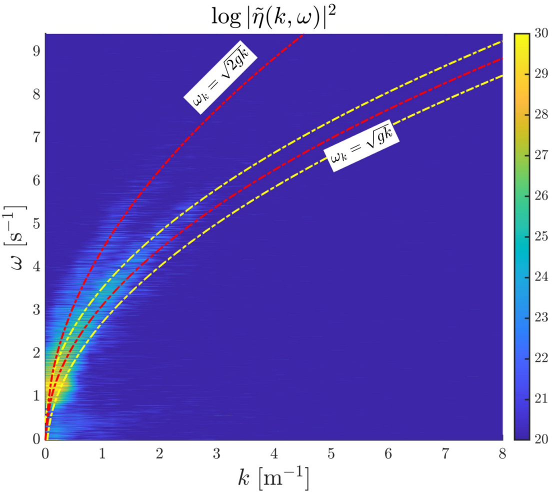


The normalized fourth-order correlator

$$\frac{|\langle \eta(\mathbf{k}_1,t)^*\eta(\mathbf{k}_2,t)^*\eta(\mathbf{k}_3,t)\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)\rangle_t|}{\langle |\eta(\mathbf{k}_1,t)||\eta(\mathbf{k}_2,t)||\eta(\mathbf{k}_3,t)||\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)|\rangle_t}$$

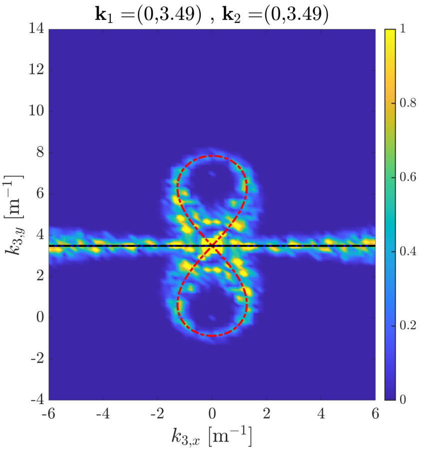


Filtering in the $(k - \omega)$ spectrum



Normalized fourth-order correlator

$$\frac{|\langle \eta^*(\mathbf{k}_1,t)\eta^*(\mathbf{k}_2,t)\eta(\mathbf{k}_3,t)\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)\rangle_t|}{\langle|\eta(\mathbf{k}_1,t)||\eta(\mathbf{k}_2,t)||\eta(\mathbf{k}_3,t)||\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)|\rangle_t}$$

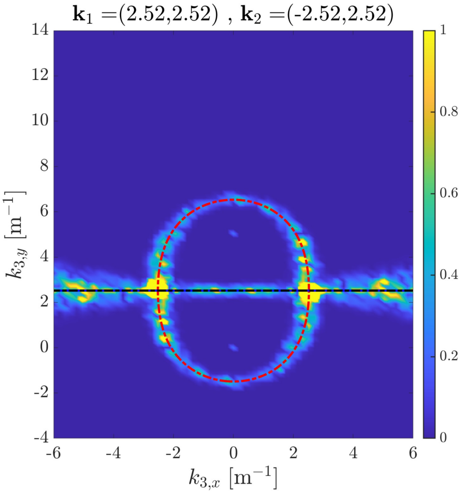


- For the black line, the following resonant condition is satisfied

$$\mathbf{k}_1 = (0, 3.49), \quad \mathbf{k}_2 = (0, 3.49),$$
$$\mathbf{k}_3 = (k_{3,x}, 3.49), \quad \mathbf{k}_4 = (-k_{3,x}, 3.49)$$
$$\omega_1 - \omega_2 = \omega_3 - \omega_4$$

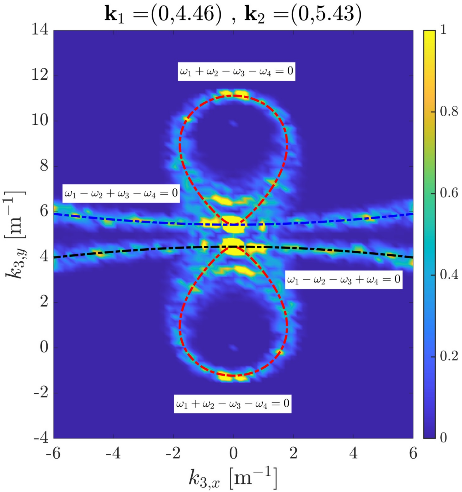
Normalized fourth-order correlator

$$\frac{|\langle \eta^*(\mathbf{k}_1,t)\eta^*(\mathbf{k}_2,t)\eta(\mathbf{k}_3,t)\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)\rangle_t|}{\langle |\eta(\mathbf{k}_1,t)||\eta(\mathbf{k}_2,t)||\eta(\mathbf{k}_3,t)||\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)|\rangle_t}$$



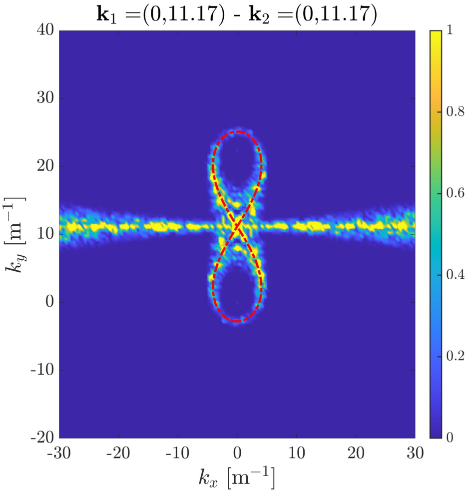
Normalized fourth-order correlator

$$\frac{|\langle \eta^*(\mathbf{k}_1,t)\eta^*(\mathbf{k}_2,t)\eta(\mathbf{k}_3,t)\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)\rangle_t|}{\langle |\eta(\mathbf{k}_1,t)||\eta(\mathbf{k}_2,t)||\eta(\mathbf{k}_3,t)||\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)|\rangle_t}$$



Data from Black-Sea

$$\frac{|\langle \eta^*(\mathbf{k}_1,t)\eta^*(\mathbf{k}_2,t)\eta(\mathbf{k}_3,t)\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)\rangle_t|}{\langle |\eta(\mathbf{k}_1,t)||\eta(\mathbf{k}_2,t)||\eta(\mathbf{k}_3,t)||\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)|\rangle_t}$$



Data from Yellow-Sea

$$\frac{|\langle \eta^*(\mathbf{k}_1,t)\eta^*(\mathbf{k}_2,t)\eta(\mathbf{k}_3,t)\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)\rangle_t|}{\langle |\eta(\mathbf{k}_1,t)||\eta(\mathbf{k}_2,t)||\eta(\mathbf{k}_3,t)||\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3,t)|\rangle_t}$$