WAVE INTERACTIONS AND TURBULENCE IN DISCRETE SYSTEMS

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WAVE INTERACTIONS AND TURBULENCE IN DISCRETE SYSTEMS

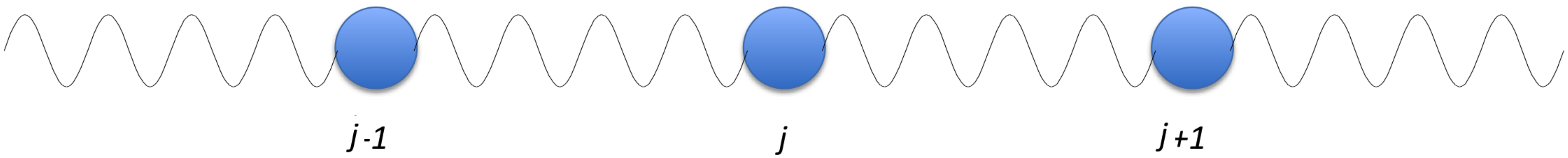
- Some results on discrete anharmonic chains (Fermi-Pasta-Ulam-Tsingou system)

  [Onorato et al, PNAS 112, 4208-4213 (2015)]
- Introduction to optical mesh lattices
- Preliminary results on thermalisation

WORK IN PROGRESS!

THE (ONE-DIMENSIONAL) ANHARMONIC CHAIN

N equal masses m connected by the same weakly nonlinear spring



$$F \simeq -\Delta q(\gamma + \alpha\Delta q + \ldots)$$

The α-FPU system has equation of motion and Hamiltonian

$$m\ddot{q}_j = \left(q_{j+1} + q_{j-1} - 2q_j\right)\left[\gamma + \alpha(q_{j+1} - q_{j-1})\right], \;\; j = 1,\ldots,N$$

$$H(p,q) = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} V(q_{j+1} - q_j)\,, \quad \text{with} \;\; V(r) = \frac{r^2}{2} + \alpha\frac{r^3}{3}$$

NORMAL MODES OF THE α-FPUT SYSTEM

Assuming periodic boundary conditions and using discrete Fourier transforms,

$$Q_k = \frac{1}{N}\sum_{j=0}^{N} q_j e^{-ijk} \qquad P_k = \dot{Q}_k$$

one obtains the wave-action variables (normal modes)

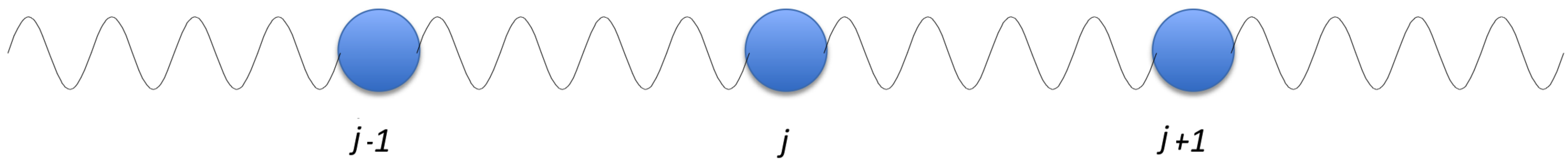
$$a_k = \frac{P_k - i\omega_k Q_k}{\sqrt{2\,\omega_k}}, \quad \text{with} \quad \omega_k = 2\left|\sin\left(\frac{k}{2}\right)\right|, k = \frac{2\pi}{N}\left(-\frac{N}{2}+1,\ldots,\frac{N}{2}\right)$$

$$\boxed{i\frac{da_1}{dt} = \omega_1 a_1 + \epsilon \sum_{k_2,k_3} V_{1,2,3}\left(a_2 a_3 \delta_{1,2+3} + 2a_2^* a_3 \delta_{1,3-2} + a_2^* a_3^* \delta_{1,-2-3}\right)}$$

$$\epsilon = \alpha\,\gamma^{1/4}/m^{3/4}\sqrt{\sum \omega_k\,|\,a_k(t_0)\,|^2}\,, \quad V_{1,2,3} = -\sqrt{\omega_1\omega_2\omega_3}/\left[2\sqrt{2}\,\mathrm{sign}(k_1k_2k_3)\right]$$

The system is Hamiltonian: $i\dfrac{da_k}{dt} = \dfrac{\delta H}{\delta a_k^*}$

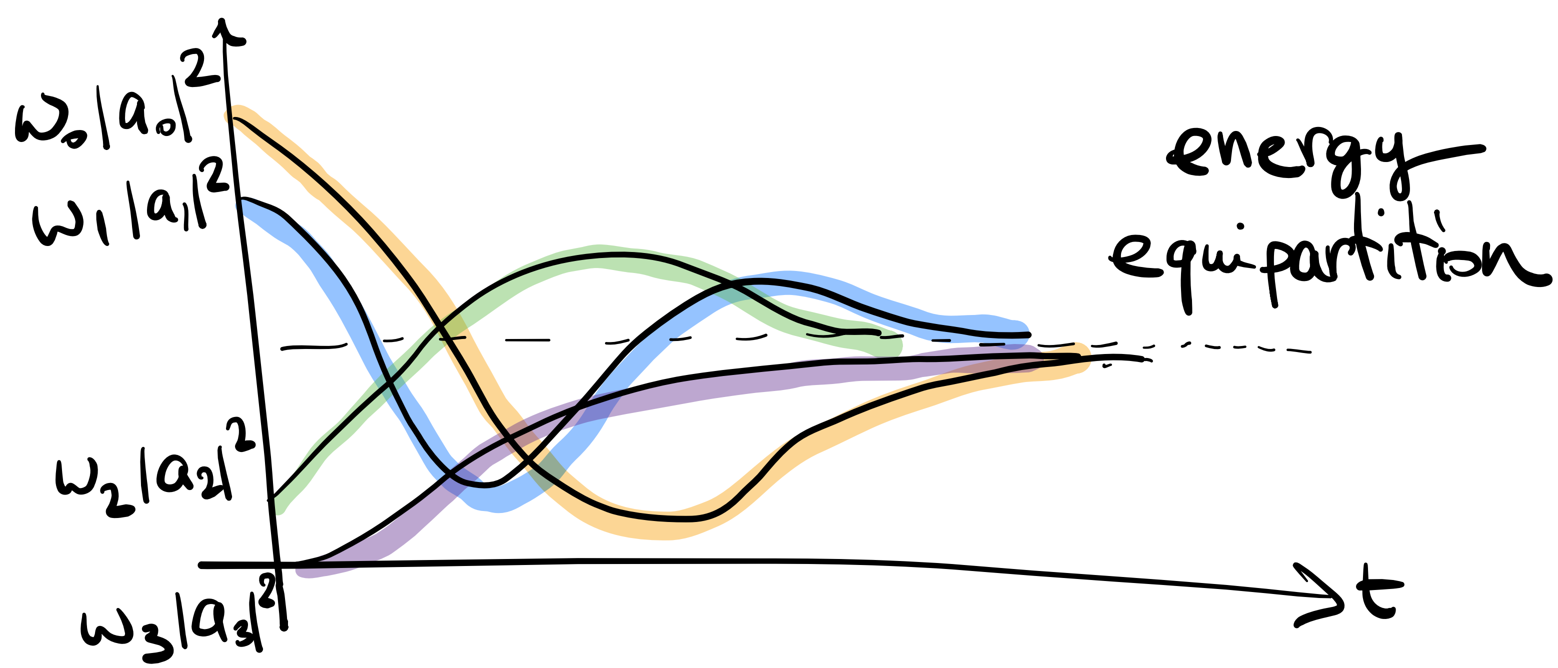
FERMI'S IDEA



Enrico Fermi was interested in modelling thermal properties and transport in solids: this was the minimal nonlinear model where to expect energy equipartition and thermalisation

$$i\frac{da_1}{dt} = \omega_1 a_1 + \epsilon \sum_{k_2,k_3} V_{1,2,3}\left(a_2 a_3 \delta_{1,2+3} + 2a_2^* a_3 \delta_{1,3-2} + a_2^* a_3^* \delta_{1,-2-3} \right)$$

Suppose
N=4



THE LOS ALAMOS REPORT

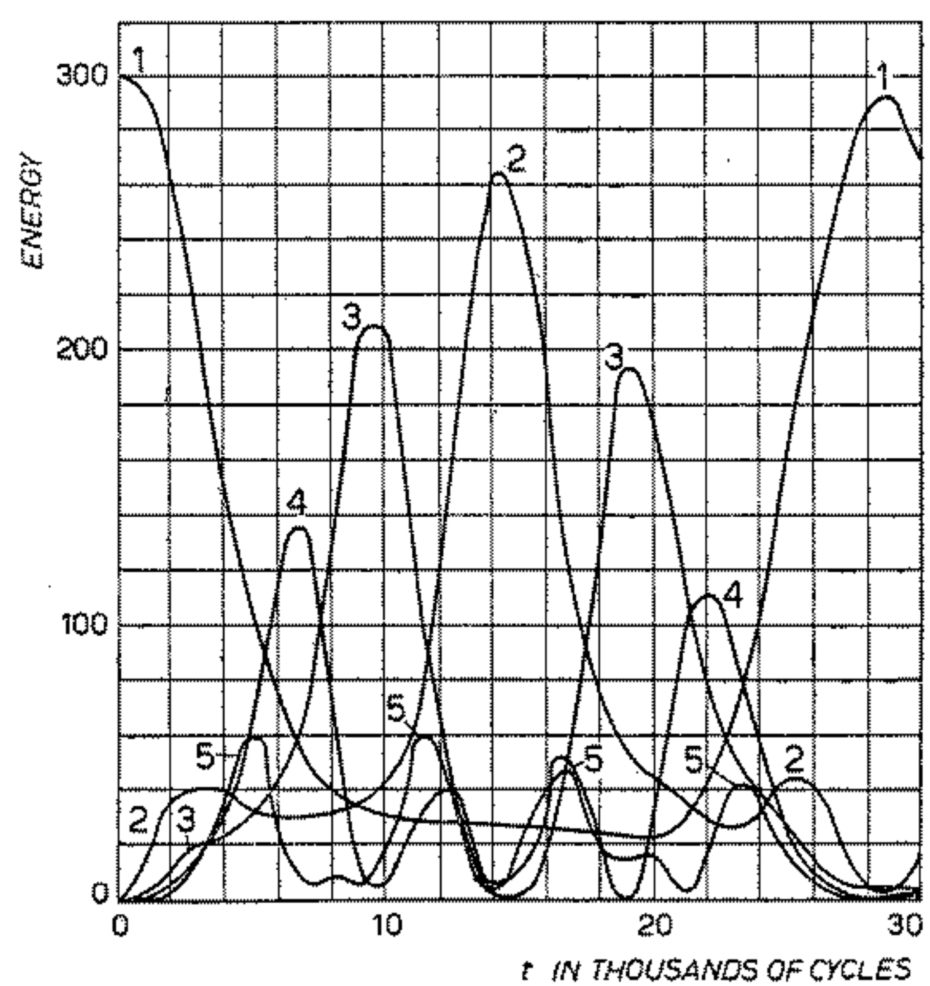
STUDIES OF NON LINEAR PROBLEMS

E. FERMI, J. PASTA, and S. ULAM

Document LA-1940 (May 1955).

A one-dimensional dynamical system of 64 particles with forces between neighbors containing nonlinear terms has been studied on the Los Alamos computer MANIAC I. The nonlinear terms considered are quadratic, cubic, and broken linear types. The results are analyzed into Fourier components and plotted as a function of time.

The results show very little, if any, tendency toward equipartition of energy among the degrees of freedom.



THE LOS ALAMOS REPORT

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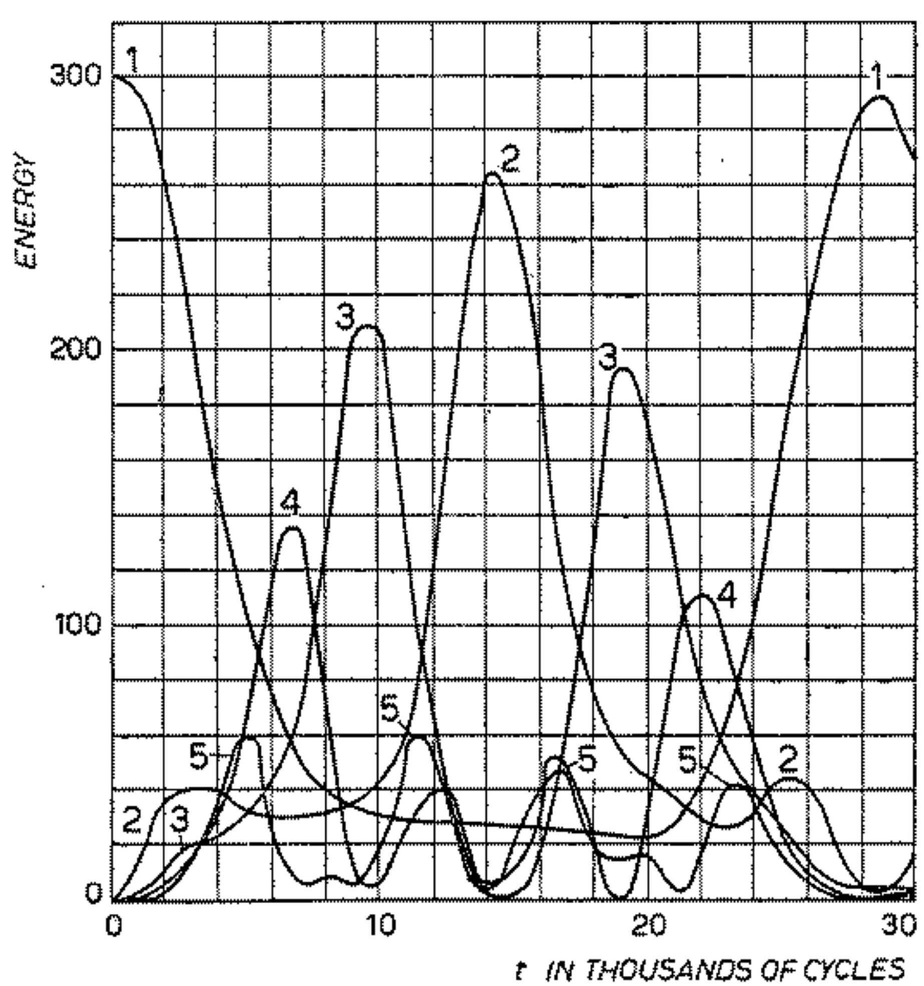
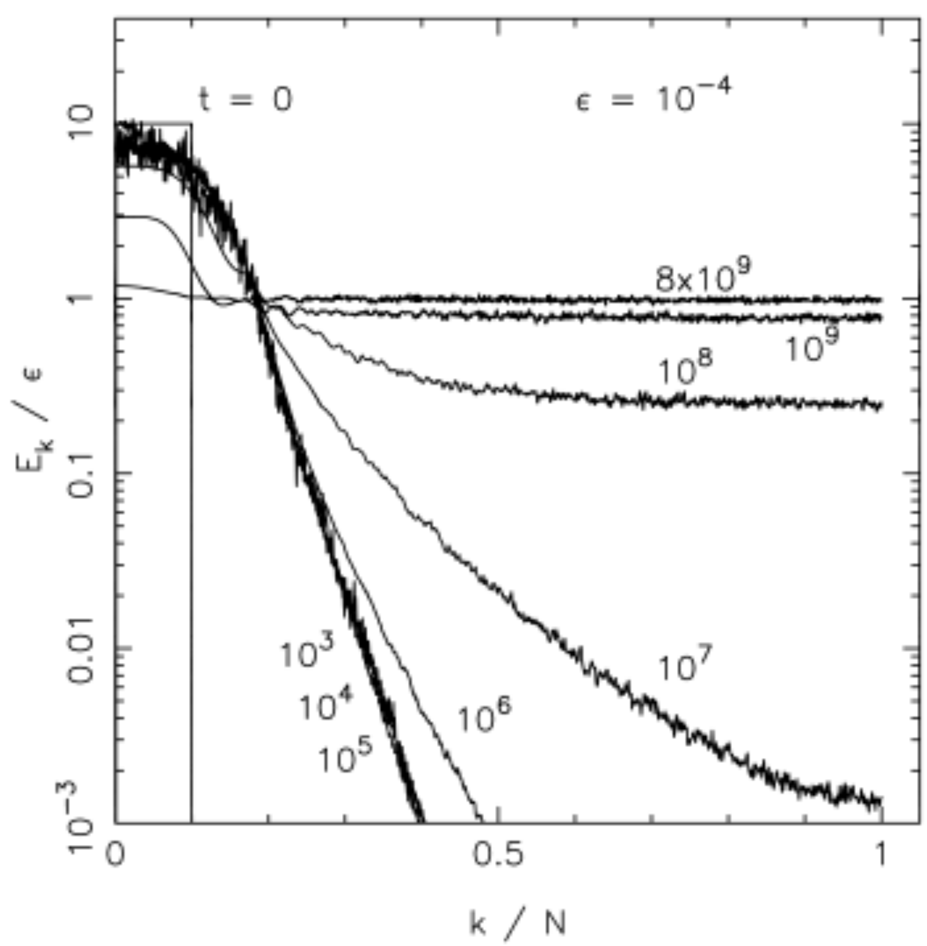




[Benettin et al., J Stat Pays 2013]

THE WAVE INTERACTION/TURBULENCE APPROACH

Wave turbulence (WT) theory is a statistical mechanics approach to weakly dispersive wave systems (waves in optics, plasma, ocean, Bose-Einstein condensates)

[Nazarenko, Wave Turbulence, 2011]

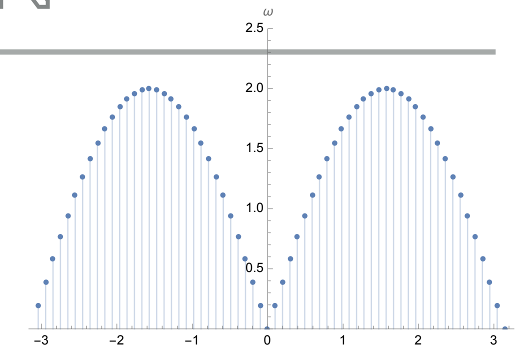
The (large time) efficient energy transfer in the system occurs only via exact resonant n-wave interaction processes satisfying

$$\boxed{\begin{aligned} k_1 \pm k_2 \pm \ldots k_n &= 0 \\ \omega_1 \pm \omega_2 \pm \ldots \omega_n &= 0 \end{aligned}}$$

(resonant manifold)

ROUTE TO EQUIPARTITION IN α-FPUT, $N = 2^l$, $l \in \mathbb{N}$

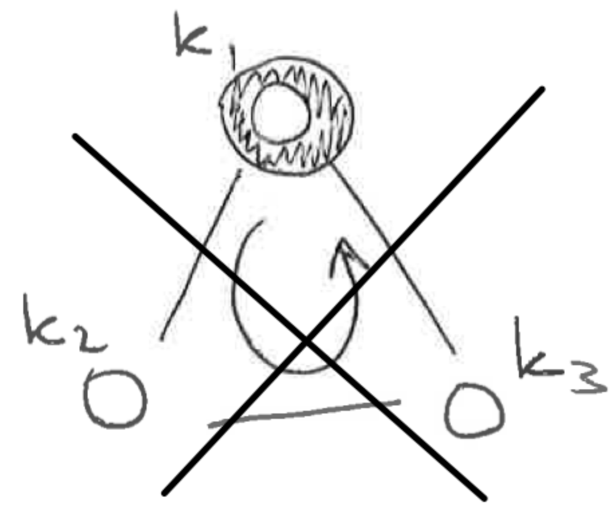
$$\boxed{\omega_k = 2\left|\sin\left(\frac{k}{2}\right)\right| \;, k = \frac{2\pi}{N}\left(-\frac{N}{2}+1,\ldots,\frac{N}{2}\right)}$$



1) No exact 3-wave interactions

$$k_1 \pm k_2 \pm k_3 \stackrel{N}{=} 0$$

$$\omega_1 \pm \omega_2 \pm \omega_3 = 0$$

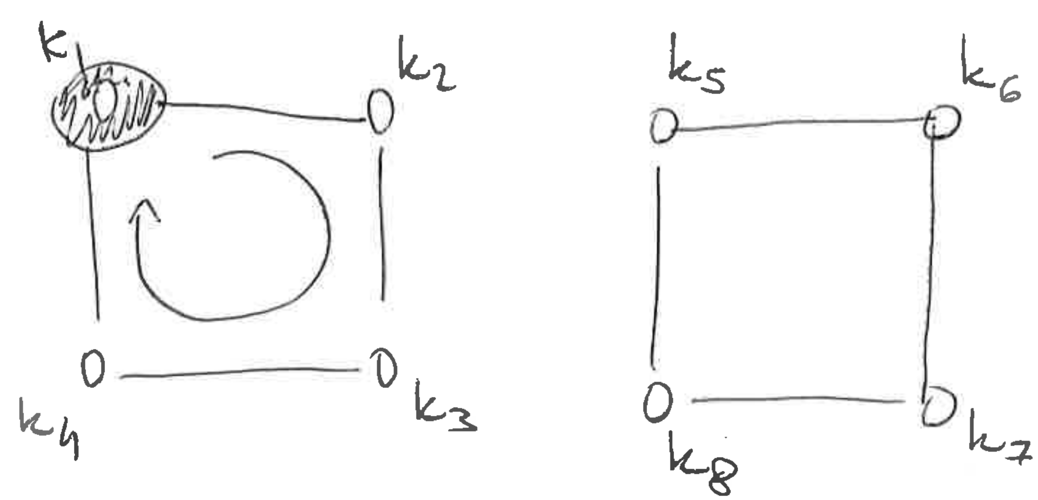


2) Canonical transformations $a_k \to b_k$, to remove the 3-wave interactions, then higher order interactions

3) 4-wave interactions are resonant, but quartets are disconnected

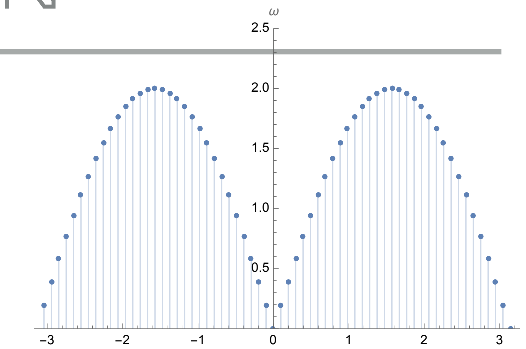
$$k_1 + k_2 - k_3 - k_4 \stackrel{N}{=} 0$$

$$\omega_1 + \omega_2 - \omega_3 - \omega_4 = 0$$



ROUTE TO EQUIPARTITION IN α-FPUT, $N = 2^l$, $l \in \mathbb{N}$

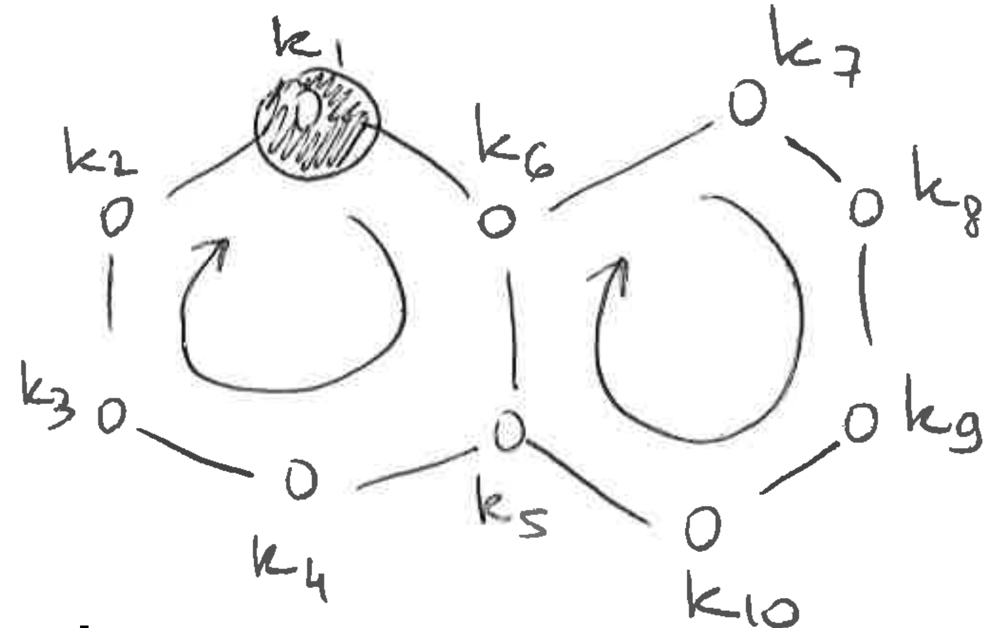
$$\boxed{\omega_k = 2\left|\sin\left(\frac{k}{2}\right)\right|, k = \frac{2\pi}{N}\left(-\frac{N}{2}+1, \dots, \frac{N}{2}\right)}$$



4) 6-wave interactions are resonant and all sextuplets are connected, <u>hence thermalisation can happen!</u>

$$k_1 + k_2 + k_3 - k_4 - k_5 - k_6 \overset{N}{=} 0$$

$$\omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6 = 0$$



5) Estimation of the equipartition time-scale

$$i\frac{db_1}{dt} = \omega_1 b_1 + \text{4-waves} + \epsilon^4 \sum W_{1,2,3,4,5,6} b_2^* b_3^* b_4 b_5 b_6 \delta_{1+2+3,4+5+6}$$

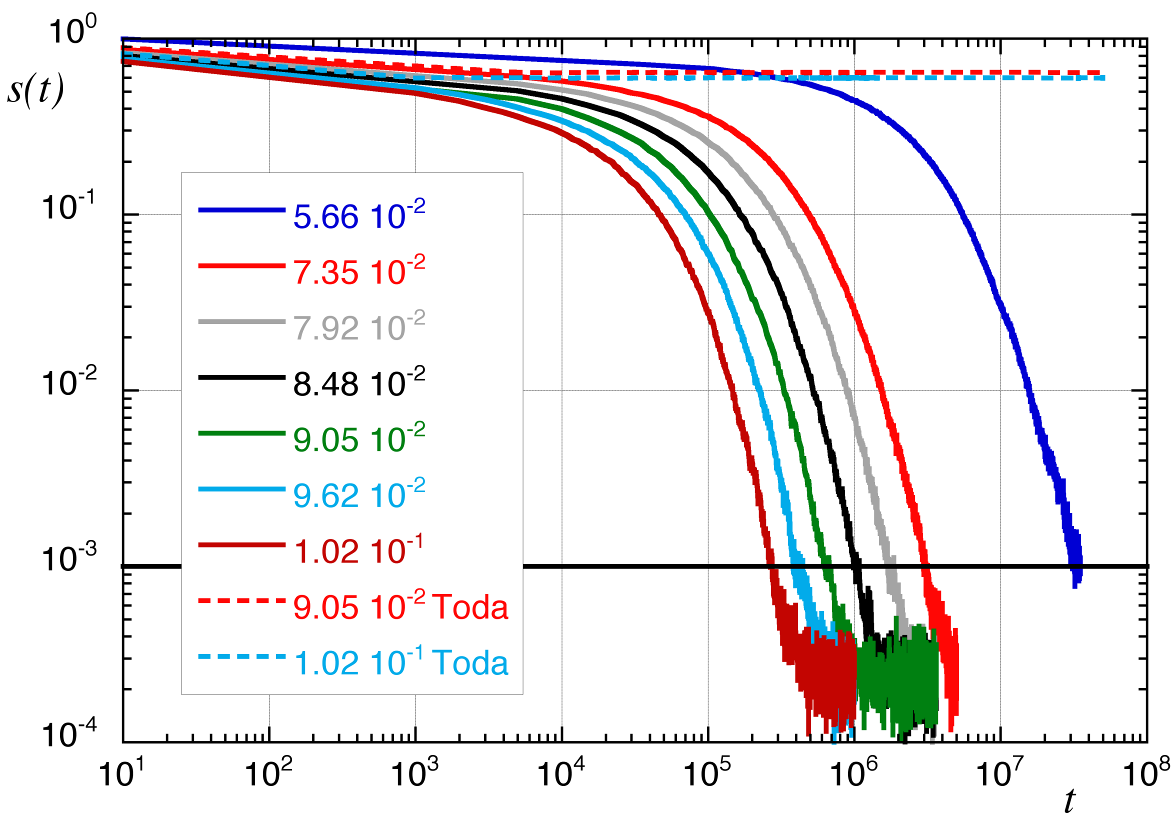
$$\frac{\partial n(k_1)}{\partial t} \sim \epsilon^8 \sum \dots \implies \boxed{t_{eq} \sim 1/\epsilon^8}$$

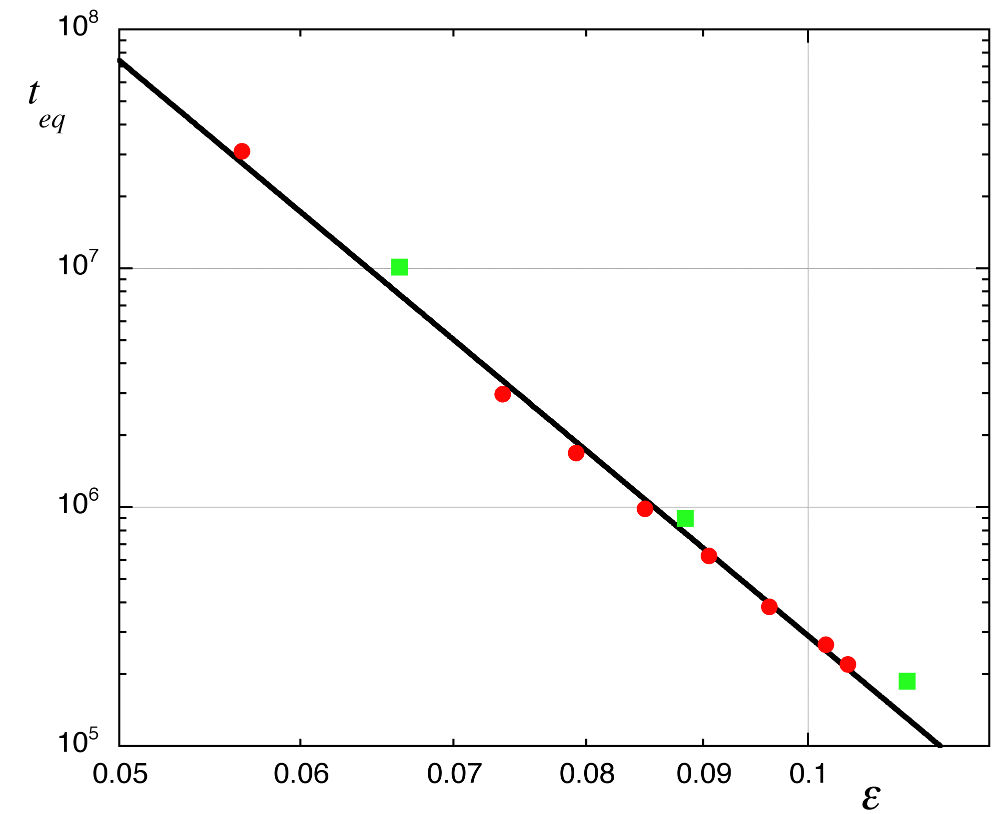
ROUTE TO EQUIPARTITION IN α-FPUT, $N = 2^l$, $l \in \mathbb{N}$

$$\frac{\partial n(k_1)}{\partial t} \sim \epsilon^8 \sum \ldots \Longrightarrow \boxed{t_{eq} \sim 1/\epsilon^8}$$

6) entropy evolution to estimate the equipartition timescale

$$s(t) = \sum_k f_k \log f_k; \quad \text{with} \quad f_k = \frac{N-1}{E_{tot}} \omega_k \langle |a_k|^2 \rangle, \quad E_{tot} = \sum_k \omega_k \langle |a_k|^2 \rangle$$





[Onorato et al, PNAS 112, 4208-4213 (2015)]

Quick summary on FPUT

- Equipartition of interacting systems, usually one-dimensional, might not be as trivial as one expects
- The WT framework is useful to predict the equipartition timescale in the weakly nonlinear regime
- In a discrete systems the resonant manifold is also discrete, and the existence of resonant modes is not enough to insure equipartition: all the modes need to be connected!

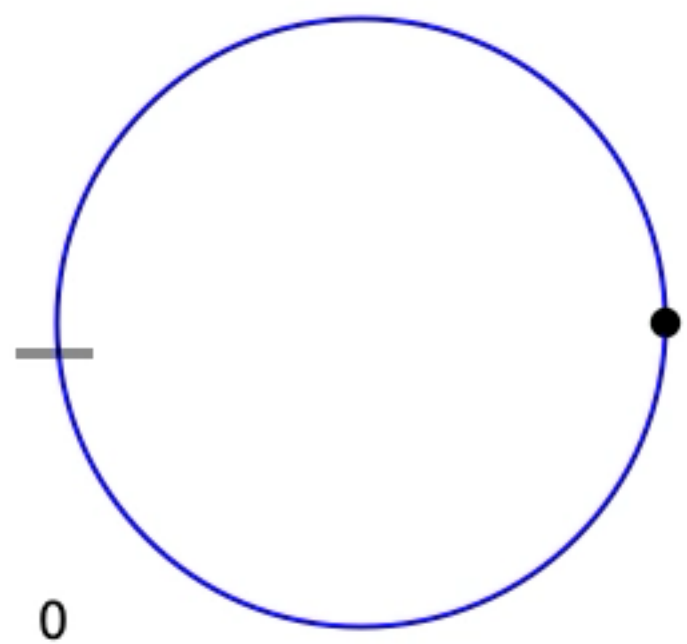
- Some results on discrete anharmonic chains (Fermi-Pasta-Ulam-Tsingou system)

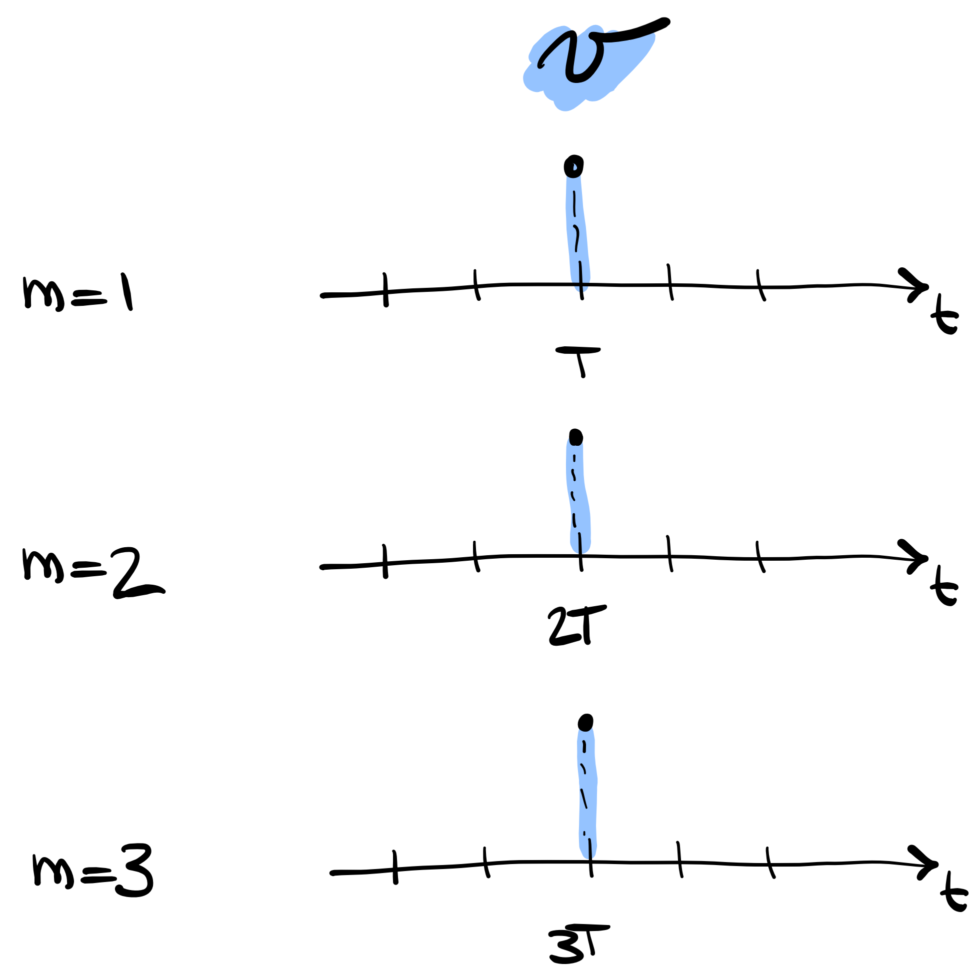
  [Onorato et al, PNAS 112, 4208-4213 (2015)]
- Introduction to optical mesh lattices
- Preliminary results on thermalisation

WORK IN PROGRESS!

OPTICAL LOOP

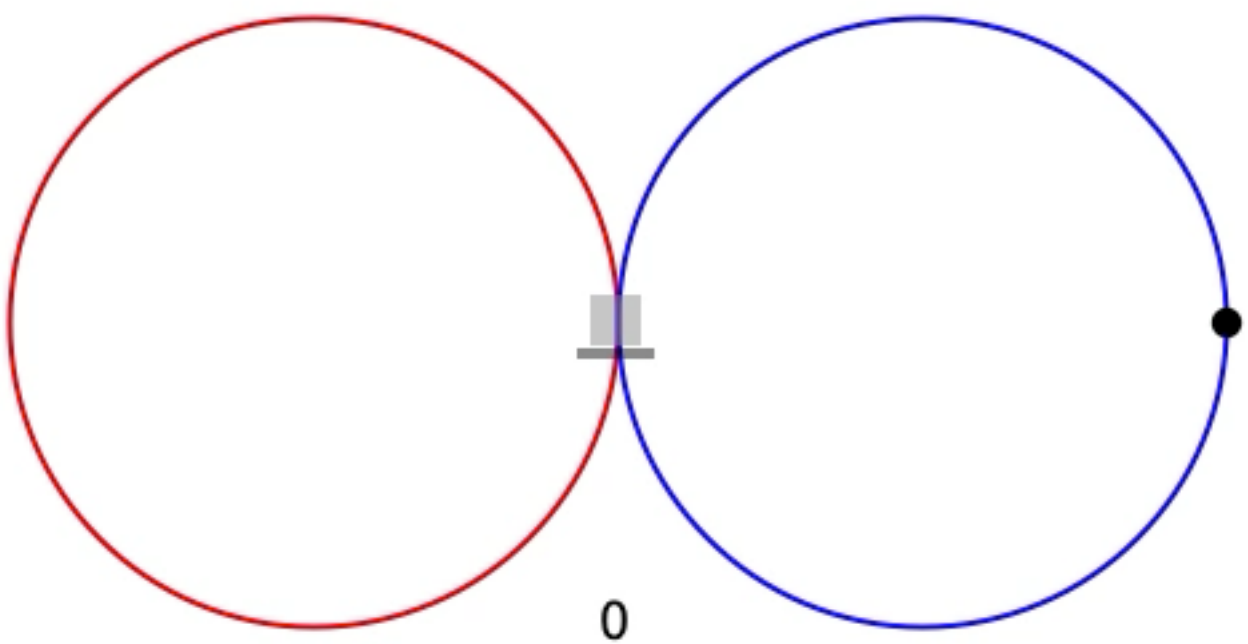
- traditional optical fibre, signal measured at straight line
- localised wave-packet (sketched as a circle)
- assume no dissipative nor dispersive effects

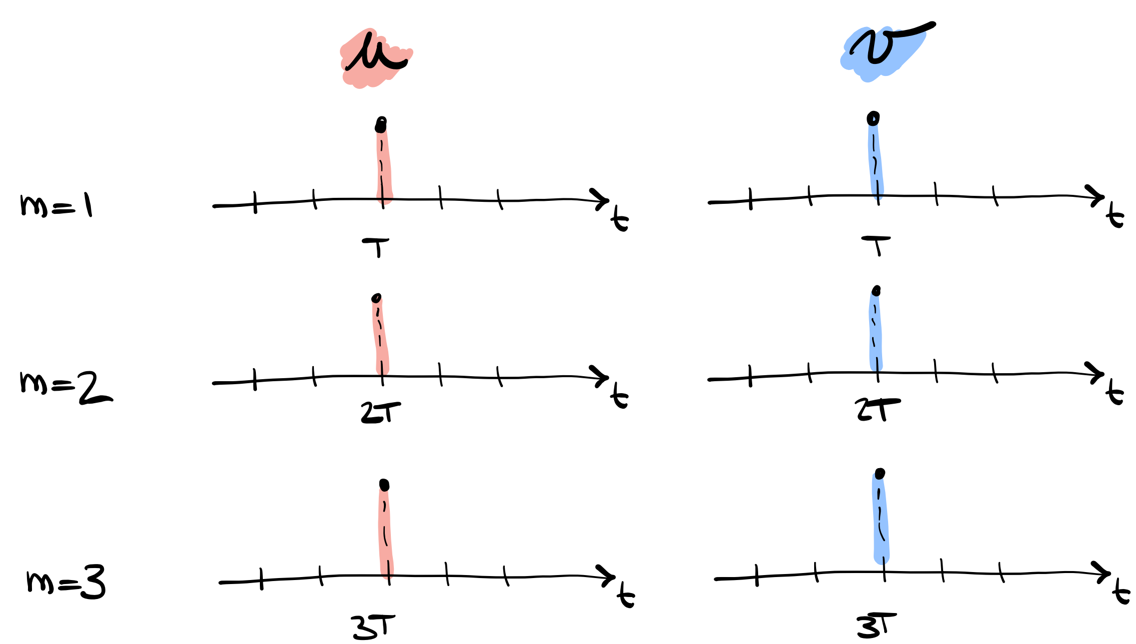




OPTICAL TWO LOOPS, EQUAL LENGTH

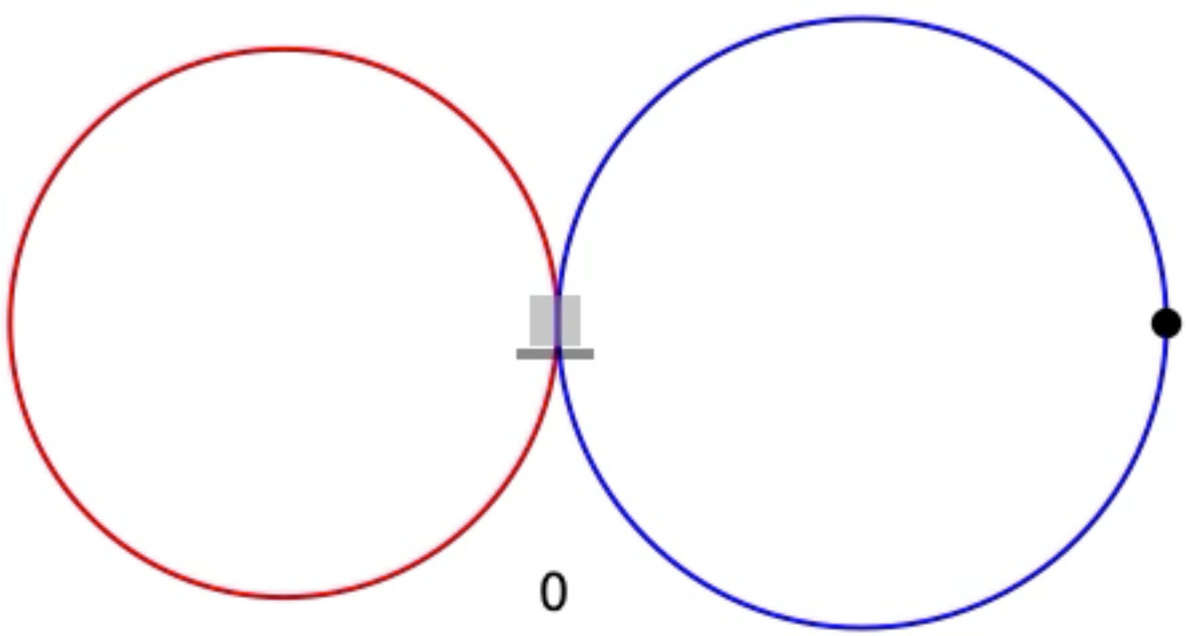
- traditional optical fibre, signal measured at straight line
- localised wave-packet (sketched as a circle)
- assume no dissipative nor dispersive effects
- two loops connected via a 50/50 beam splitter: double loop

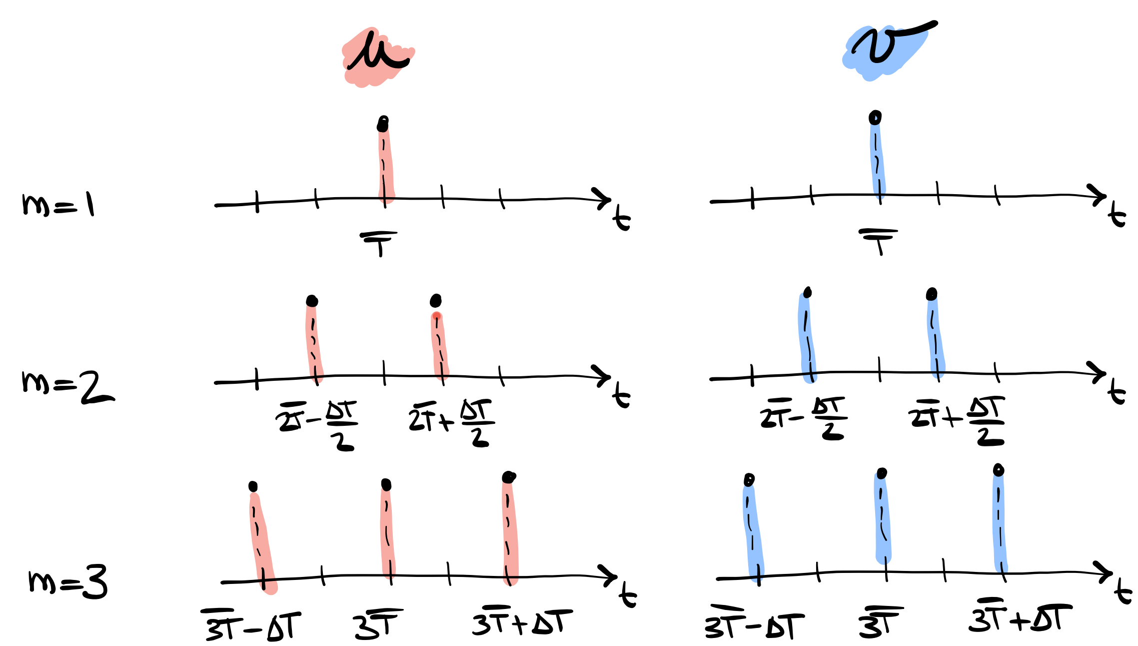




OPTICAL TWO LOOPS, DIFFERENT LENGTHS

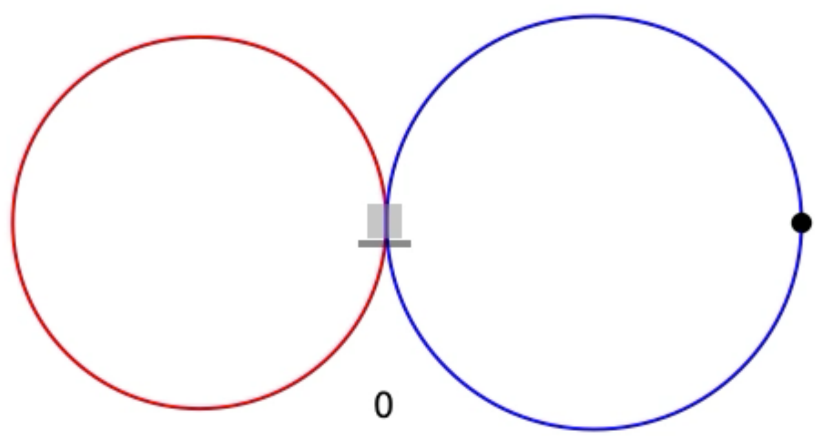
- traditional optical fibre, signal measured at straight line
- localised wave-packet (sketched as a circle)
- assume no dissipative nor dispersive effects
- two loops connected via a 50/50 beam splitter
- the loops have (slightly) different lengths

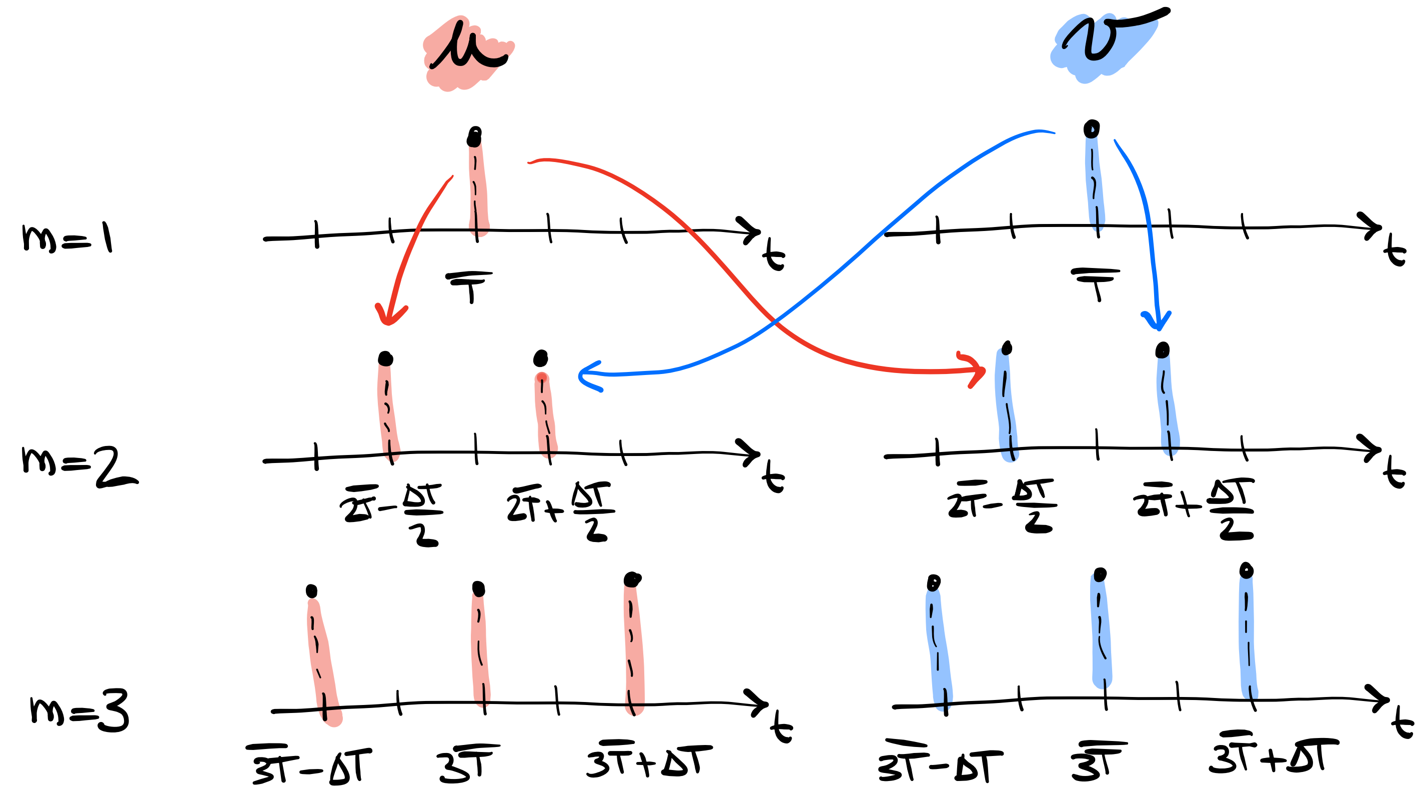




$$\bar{T} = \frac{T_1 + T_2}{2} \quad \text{and} \quad \Delta T = |T_1 - T_2|$$

THE MATHEMATICAL (LINEAR) MODEL





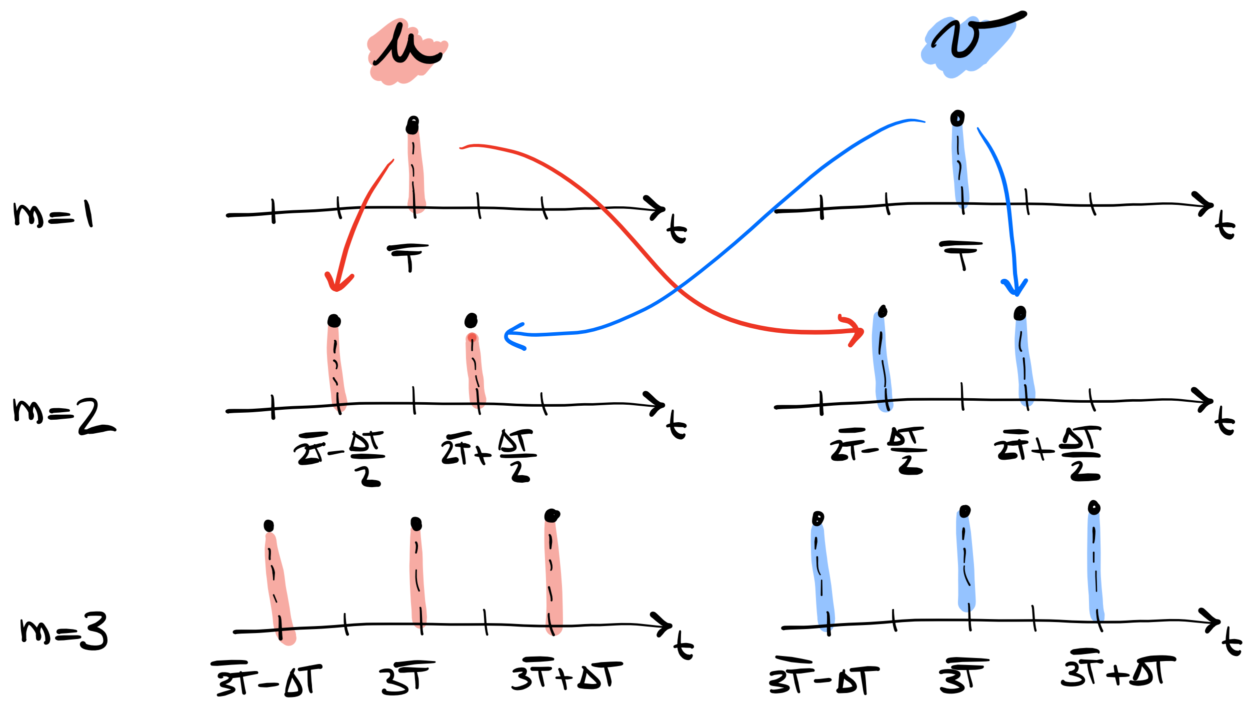
$$u_n^{m+1}\left(u_{n+1}^m,\ v_{n-1}^m\right) \longrightarrow u_n^{m+1}\left(u_{n+1}^m,\ v_{n+1}^m\right)$$

$$v_n^{m+1}\left(u_{n+1}^m,\ v_{n-1}^m\right)$$

relabeling of v lattice index, $n \to n+2$

$$v_{n+2}^{m+1}\left(u_{n+1}^m,\ v_{n+1}^m\right) \longrightarrow v_n^{m+1}\left(u_{n-1}^m,\ v_{n-1}^m\right)$$

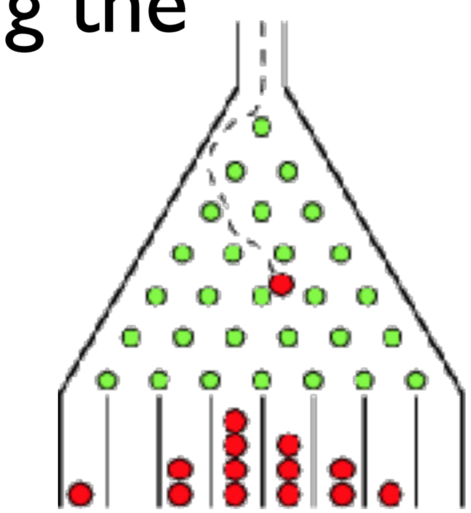
THE MATHEMATICAL (LINEAR) MODEL



$$u_n^{m+1} = \frac{1}{\sqrt{2}}\left(u_{n+1}^m + i v_{n+1}^m\right)$$

$$v_n^{m+1} = \frac{1}{\sqrt{2}}\left(i u_{n-1}^m + v_{n-1}^m\right)$$

- The signals are complex, $(u_n^m, v_n^m) \in \mathbb{C}$, with (n, m) being the spatial and time lattice indices, respectively
- The 50/50 beam splitter introduce an $\pi/2$ phase in the reflected signal
- System equivalent to an (averaged) Galton board
- Signals on even (odd) spatial lattice points will come back to even (odd) spatial lattice points after two steps



[Experiments on pulse dynamics and parity-time symmetry in optical fiber networks, Alois Regensburger, PhD dissertation (2013)]

TWO-STEPS DYNAMICS, BLOCH-FLOQUET ANSATZ

$$u_n^{m+1} = \frac{1}{\sqrt{2}}\left(u_{n+1}^m + i v_{n+1}^m\right)$$
$$v_n^{m+1} = \frac{1}{\sqrt{2}}\left(i u_{n-1}^m + v_{n-1}^m\right)$$

$$\Longrightarrow$$

$$\boxed{\begin{aligned} u_n^{m+2} &= \frac{1}{2}\left(u_{n+2}^m - u_n^m + i v_{n+2}^m + i v_n^m\right) \\ v_n^{m+2} &= \frac{1}{2}\left(i u_{n-2}^m + i u_{n-2}^m - v_n^m + v_{n-2}^m\right) \end{aligned}}$$

- Two steps in space and time, $n' = 2n, m' = 2m, N' = N/2$
- With Bloch-Floquet ansatz

$$\begin{pmatrix} \tilde{u}_k^{m'} \\ \tilde{v}_k^{m'} \end{pmatrix} = \sum_{n'=0}^{N'-1} \begin{pmatrix} u_{n'}^{m'} \\ v_{n'}^{m'} \end{pmatrix} e^{-ikn'} e^{i\omega m'} \quad \Longrightarrow$$

$$\omega_k^{\pm} = \pm \arccos\left[\frac{1}{2}\cos\left(-\frac{k}{2}\right) - \frac{1}{2}\right] + 2\pi l$$

with $\quad k = \frac{2\pi}{N'}\left(-\frac{N'}{2} + 1, \ldots, \frac{N'}{2}\right), \quad l \in \mathbb{Z}$

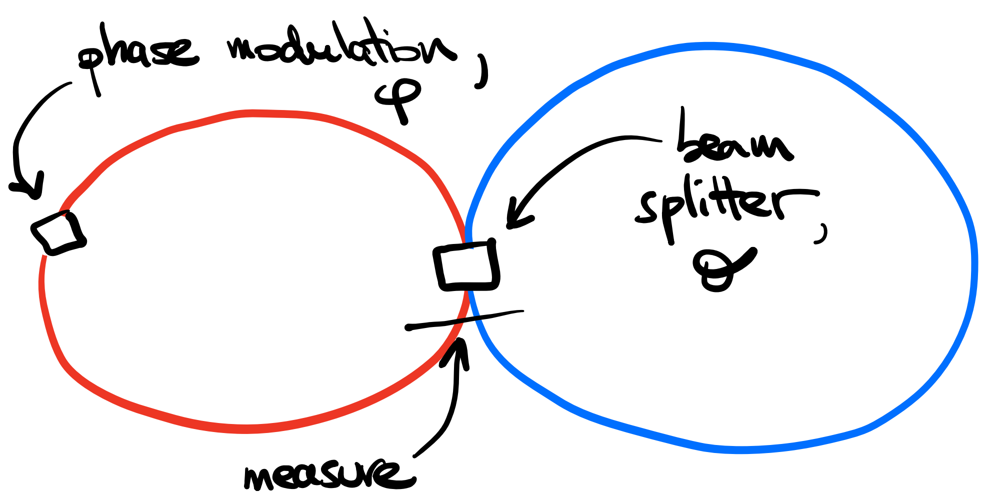
- Two opposite periodic branches
- Normal modes are not simply the $(\tilde{u}_k^{m'}, \tilde{v}_k^{m'})$, but requires a diagonalisation procedure

$$\begin{pmatrix} \tilde{a}_k^{m',+} \\ \tilde{a}_k^{m',-} \end{pmatrix} = D \begin{pmatrix} \tilde{u}_k^{m'} \\ \tilde{v}_k^{m'} \end{pmatrix}$$

THE MATHEMATICAL (LINEAR) MODEL

A more general (linear) system

- beam splitter with variable slitting ratio, parameter θ^m
- Variable phase modulation in one of the loop, parameter ϕ^m



$$u_n^{m+1} = \left[\cos(\theta^m) u_{n+1}^m + i \sin(\theta^m) v_{n+1}^m\right] e^{i\phi^m}$$
$$v_n^{m+1} = i \sin(\theta^m) u_{n-1}^m + \cos(\theta^m) v_{n-1}^m$$

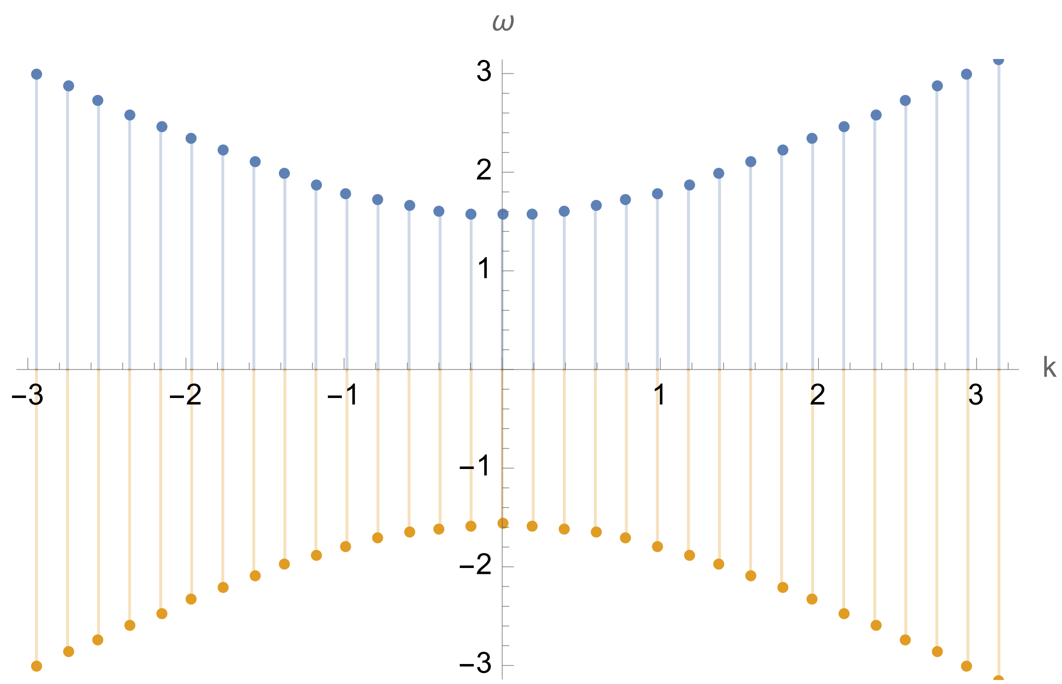
Assuming an alternating phase modulation $\phi^m = \begin{cases} -\phi & \text{for } m \text{ even} \\ \phi & \text{for } m \text{ odd} \end{cases}$
the dispersion relation results in

$$\omega_k^{\pm} = \pm \arccos\left[\cos^2(\theta) \cos\left(-\frac{k}{2}\right) - \sin^2(\theta) \cos^2(\phi)\right] + 2\pi l$$

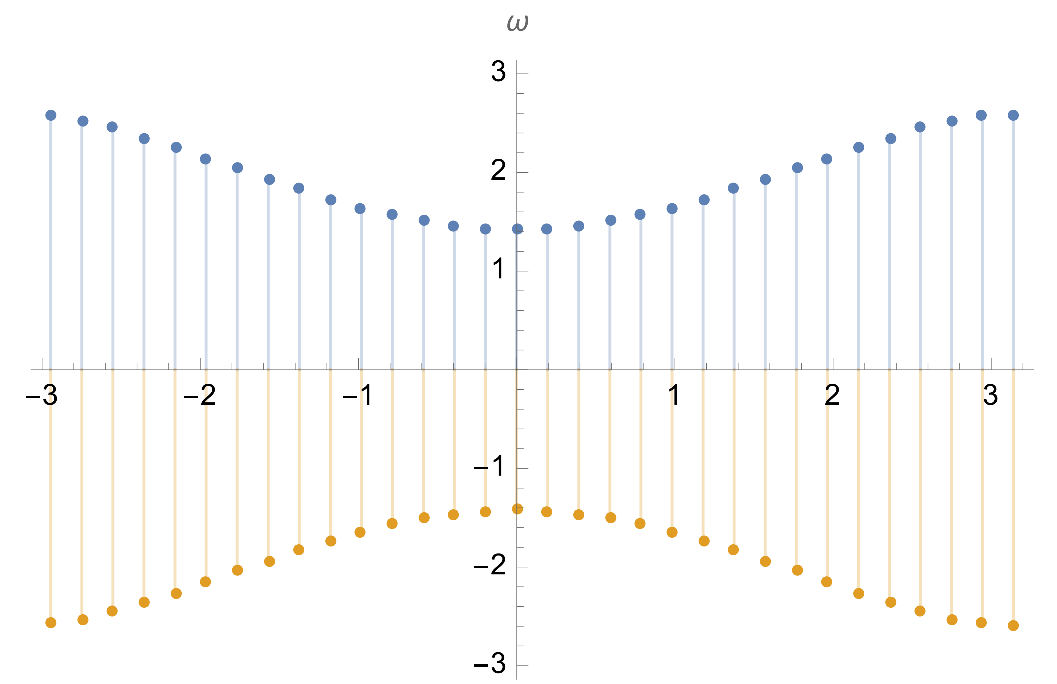
with $\quad k = \frac{2\pi}{N'}\left(-\frac{N'}{2} + 1, \ldots, \frac{N'}{2}\right) \quad$ and $\quad l \in \mathbb{Z}$

[L.K. Upreti,et al, PRL 125, 186804 (2020)]

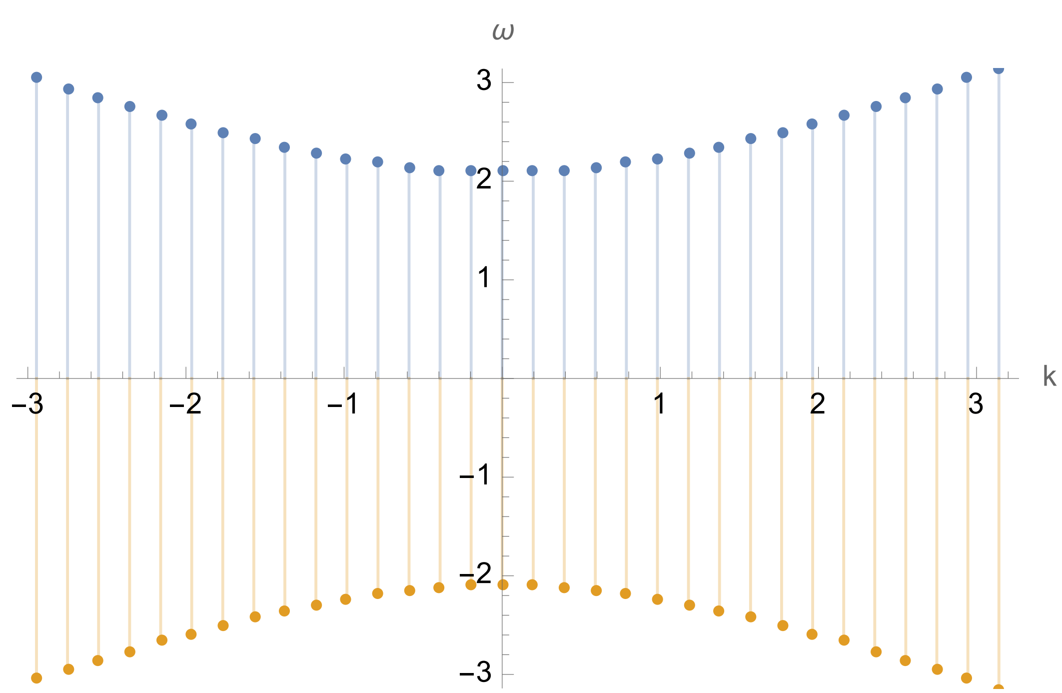
ENGINEERING THE DISPERSION RELATION, $N' = 32$



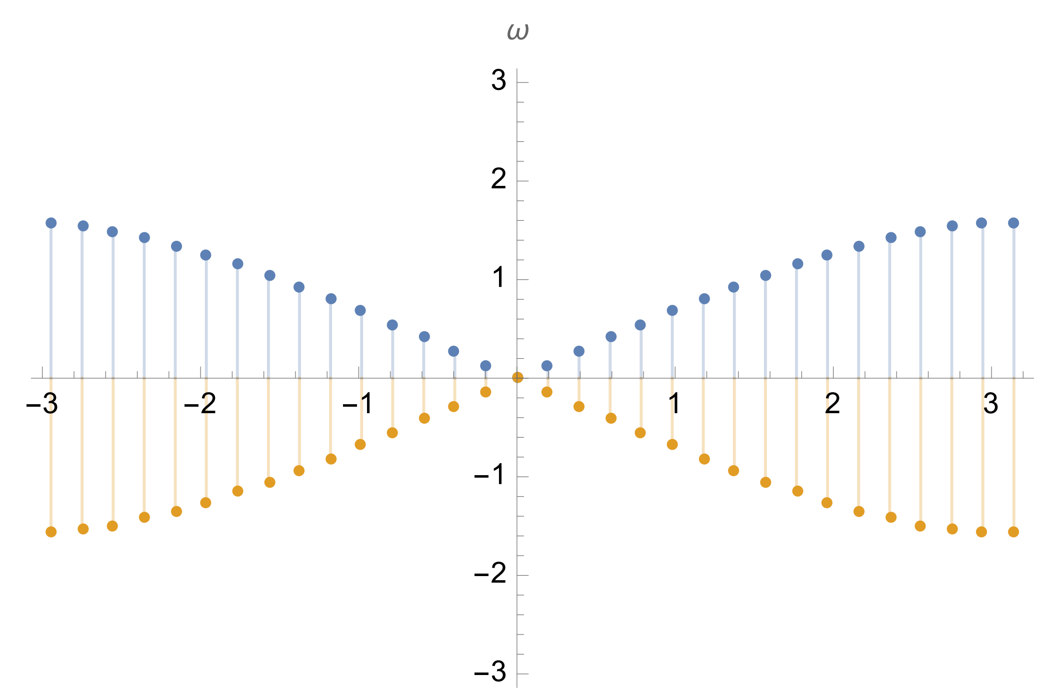
$\theta = \pi/4\,, \quad \phi = 0$



$\theta = \pi/4\,, \quad \phi = \pi/4$

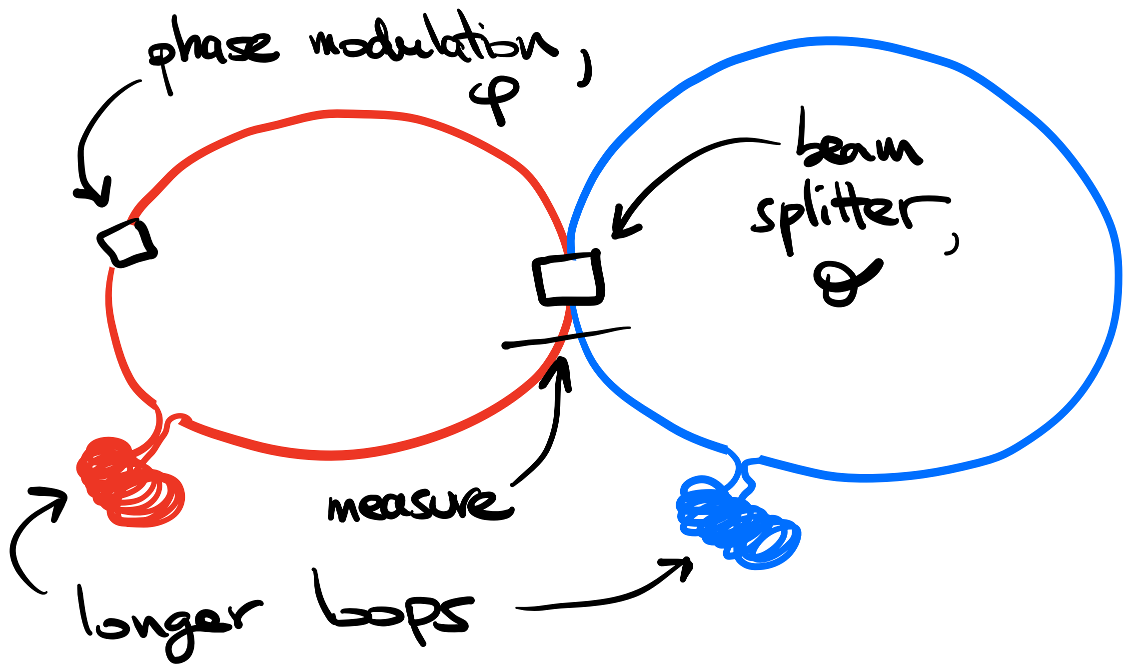


$\theta = \pi/3\,, \quad \phi = 0$



$\theta = \pi/4\,, \quad \phi = \pi$

ADDING NONLINEAR INTERACTIONS



from NLS, assuming no dispersive effects:

$$\psi(t) = \psi(t_0)e^{i|\psi(t_0)|^2(t-t_0)}$$

- take long loops / high powers so that nonlinear effects in the signal propagation need to be considered
- while making a loop each signal acquires an extra phase rotation proportional to a nonlinear parameter χ times its absolute value squared

$$u_n^{m+1} = \left[\cos(\theta^m)u_{n+1}^m e^{i\chi|u_{n+1}^m|^2} + i\sin(\theta^m)v_{n+1}^m e^{i\chi|v_{n+1}^m|^2}\right] e^{i\phi^m}$$

$$v_n^{m+1} = i\sin(\theta^m)u_{n-1}^m e^{i\chi|u_{n-1}^m|^2} + \cos(\theta^m)v_{n-1}^m e^{i\chi|v_{n-1}^m|^2}$$

[Experiments on pulse dynamics and parity-time symmetry in optical fiber networks, Alois Regensburger, PhD dissertation (2013)]

- thermal states have been measured experimentally and explained using statistical mechanics principles

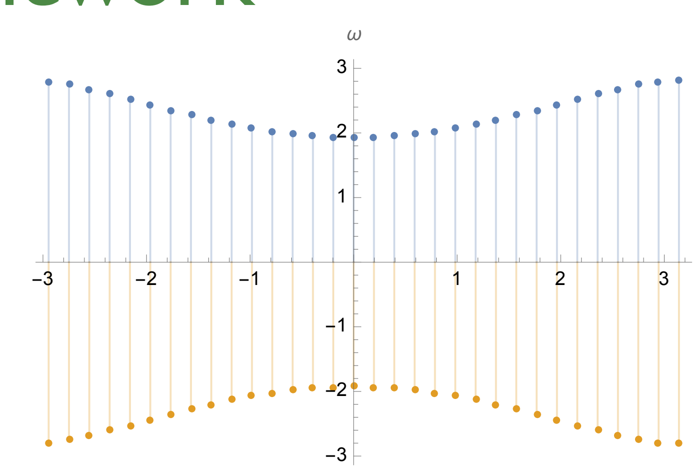
Marques Muniz *et al.*, *Science* **379**, 1019–1023 (2023)

THERMODYNAMICS

Observation of photon-photon thermodynamic processes under negative optical temperature conditions

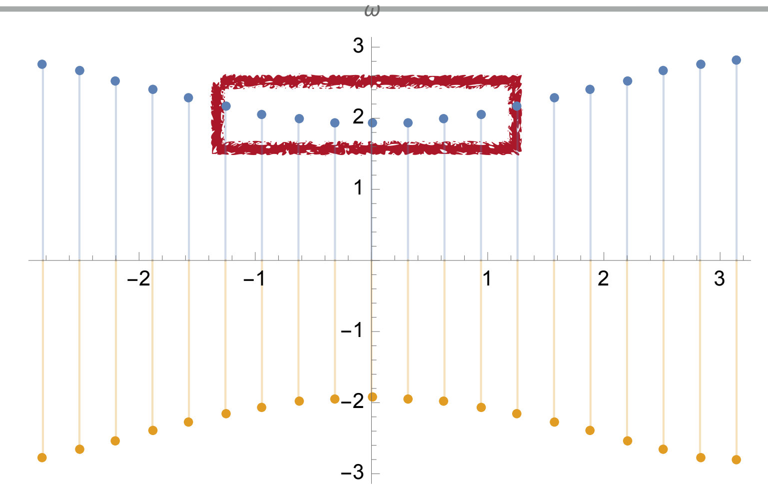
A. L. Marques Muniz[1,2]†, F. O. Wu[3]†, P. S. Jung[3,4]†, M. Khajavikhan[5], D. N. Christodoulides[3,5]*, U. Peschel[1]*

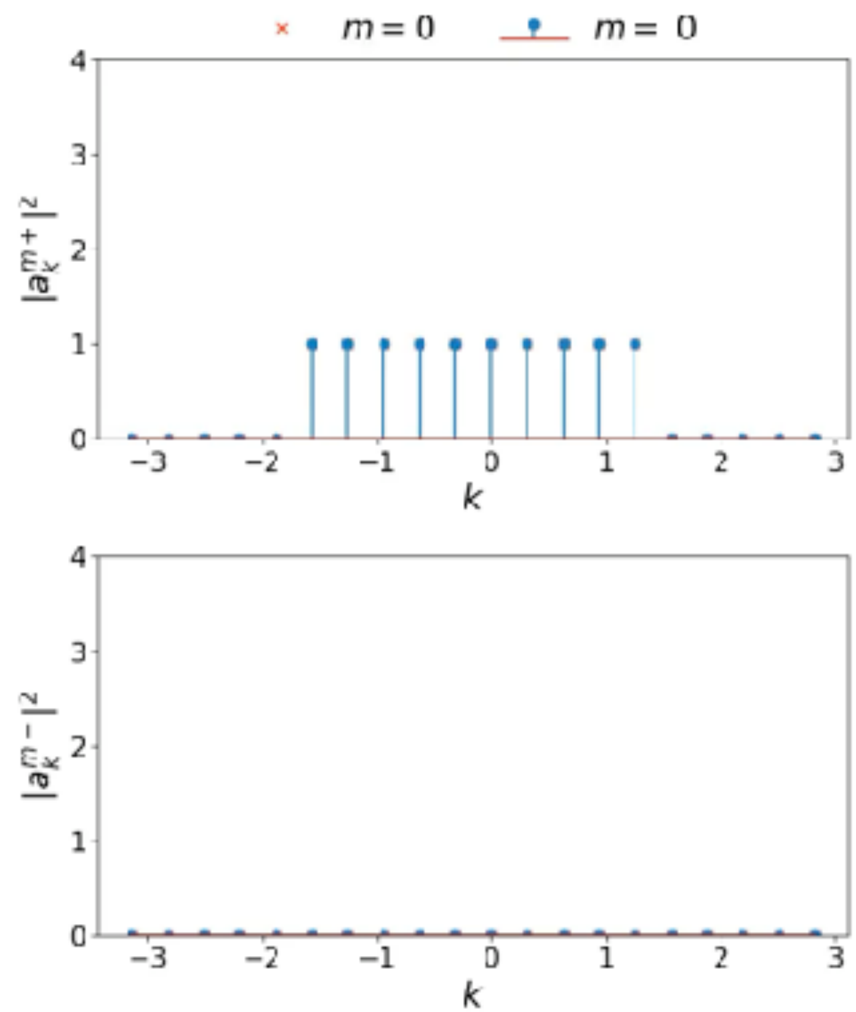
- we want to characterise the wave interaction processes and the route to thermalisation using the WT framework
- we choose $\theta = \arccos\left(\sqrt{0.3}\right)$ and $\phi = \pi/8$



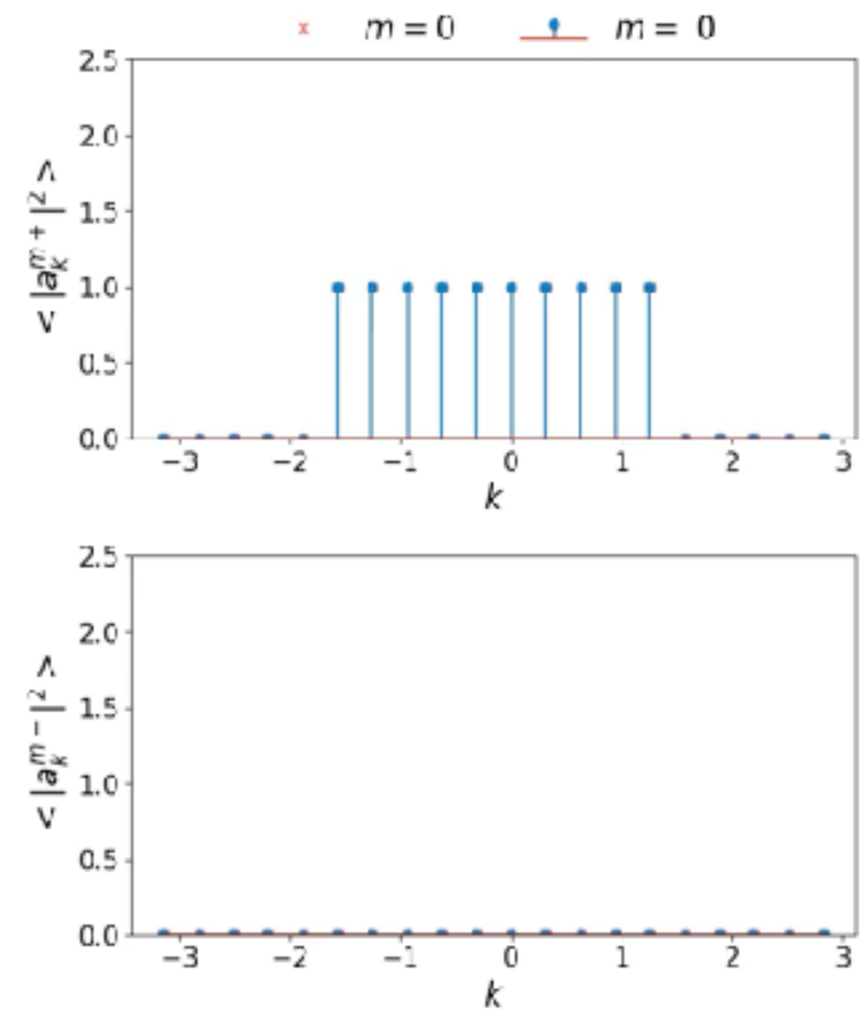
THERMALISATION - HIGHER BRANCH

- Lowest $N'/2 = 20$ energy modes of the upper branch, $\theta = \arccos\left(\sqrt{0.3}\right)$ and $\phi = \pi/8$





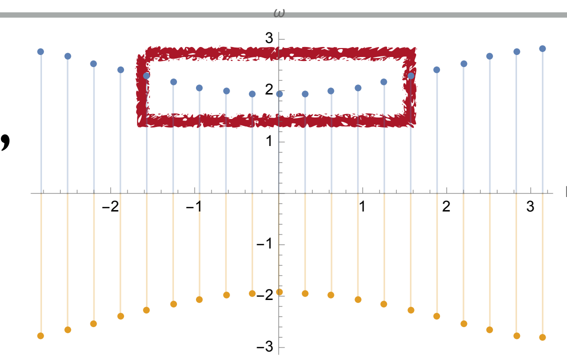
a single realisation



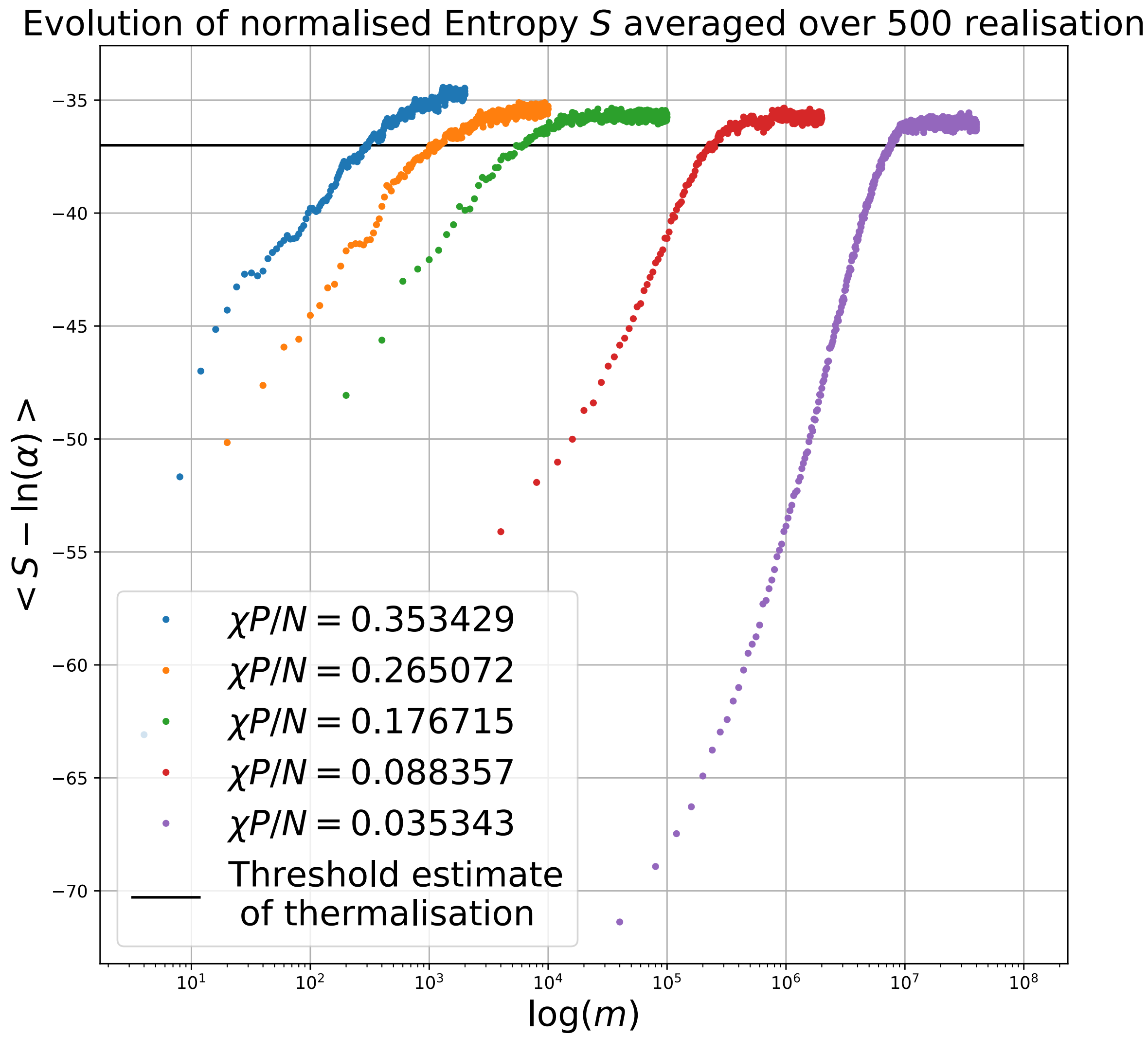
average over 500 realisations

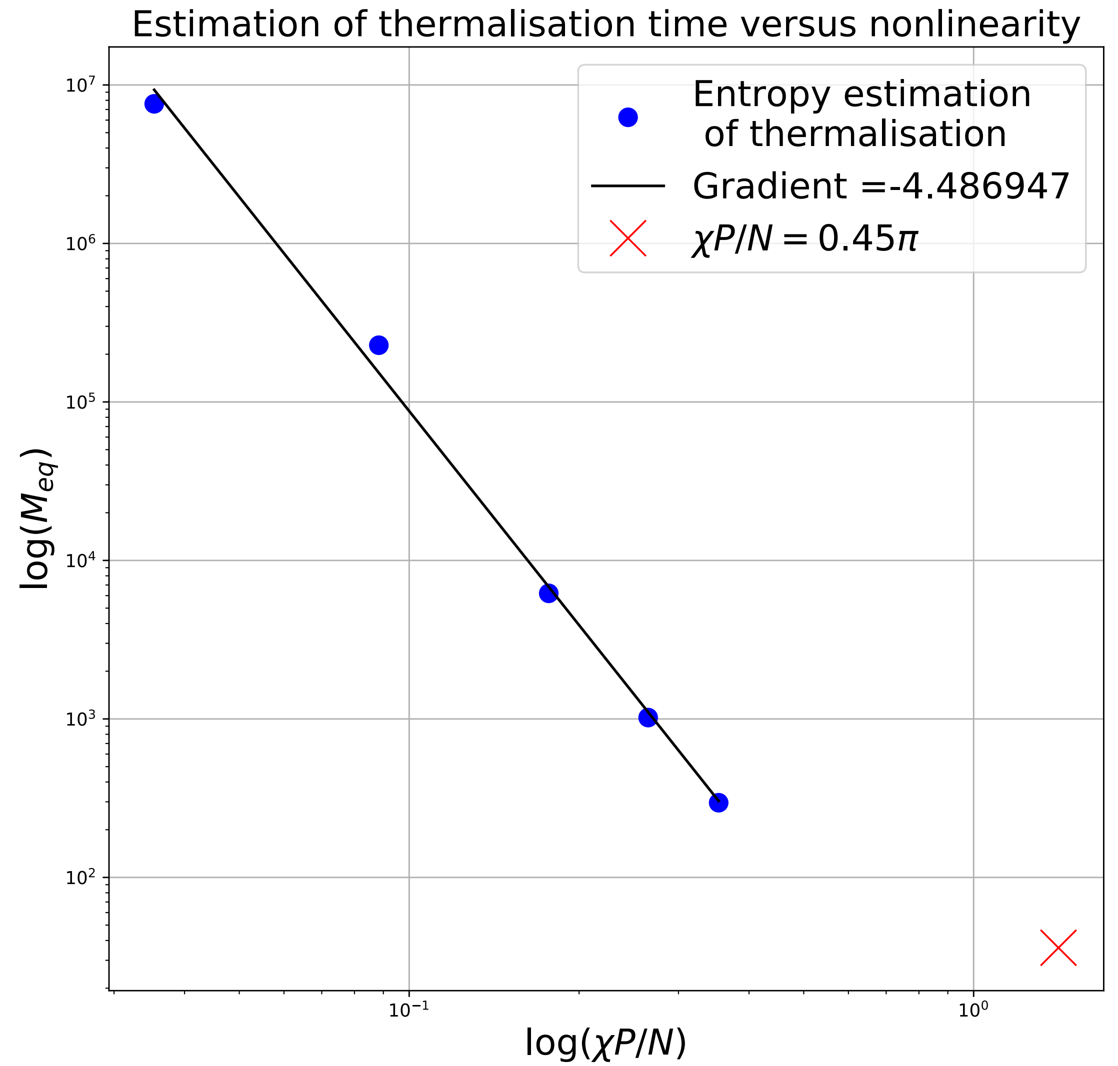
ENTROPY EVOLUTION AND EQUIPARTITION TIMESCALE

- Lowest $N'/2 = 20$ energy modes of the upper branch, $\theta = \arccos\left(\sqrt{0.3}\right)$ and $\phi = \pi/8$



$$s^m = \sum_k \log \frac{|a_k^{m,+}|^2 + |a_k^{m,-}|^2}{P}, \quad P = \sum_k \left(|a_k^{m,+}|^2 + |a_k^{m,-}|^2 \right) = \text{const.}$$





PRELIMINARY CONCLUSIONS

$$u_n^{m+1} = \left[\cos(\theta^m) u_{n+1}^m e^{i\chi |u_{n+1}^m|^2} + i \sin(\theta^m) v_{n+1}^m e^{i\chi |v_{n+1}^m|^2}\right] e^{i\phi^m}$$

$$v_n^{m+1} = i \sin(\theta^m) u_{n-1}^m e^{i\chi |u_{n-1}^m|^2} + \cos(\theta^m) v_{n-1}^m e^{i\chi |v_{n-1}^m|^2}$$

- Flexible system allowing to engineer the dispersion relation
- Thermalisation seems to be dominated by 4-wave process (see Tommy's poster!)
- (T, μ) fit still missing! Not clear how both branches thermalise

<u>Open questions</u>

- Derivation of a discrete kinetic equation? Is the time discreteness playing an important role?
- Out-of-equilibrium steady states? Turbulence?

WORK IN PROGRESS

THANKS FOR YOUR ATTENTION!

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