Happy birthday Sergey!

Congratulations

Some thoughts on finite-time singularity solutions of the Euler equations

S Rica, PHYSICAL REVIEW FLUIDS 7, 034401 (2022)
R Cadiz, D Martinez-Arguello, S Rica, Adv Cont Discr Mod 2023, 30 (2023).
MSc Thesis D Martinez-Arguello (2023)
D Martinez-Arguello & S Rica, to appear in PRFluids (2024)

#Supported by Fondecyt 1220369 (ANID)

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Les Houches, Sep 2nd, 2024

Outline

- Some history and perspective
- For Euler equations the existence of singularities may depend on initial conditions, the geometry, etc.
- Finite time singularities in other PDEs.

- Evidence of a self-similar pointlike singularity of solutions of Euler equations.
- Discussion.

The Euler equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p$$

$$\nabla \cdot \mathbf{v} = 0$$

Euler 1757 (2nd PDE written in history, excellent description of perfect fluids)

- **Symmetries:** translation, rotations, scaling, galilean invariance.
- **Conserved quantities:** If the velocity field is differentiable, then the Energy $E = \frac{1}{2}\int |\mathbf{v}|^2 d^3x < \infty$, the Circulation (Kelvin theorem), the Helicity, the Linear Momentum, the mean vorticity are conserved by the flow.
- **Taxonomy:** Euler (as well as Navier-Stokes) equations are hard because are a nonlinear and **non-local** PDE (Same for vorticity equation).
- **Timely:** Recent attention to the Regularity problem: Does a smooth initial condition remain differentiable for all time ?

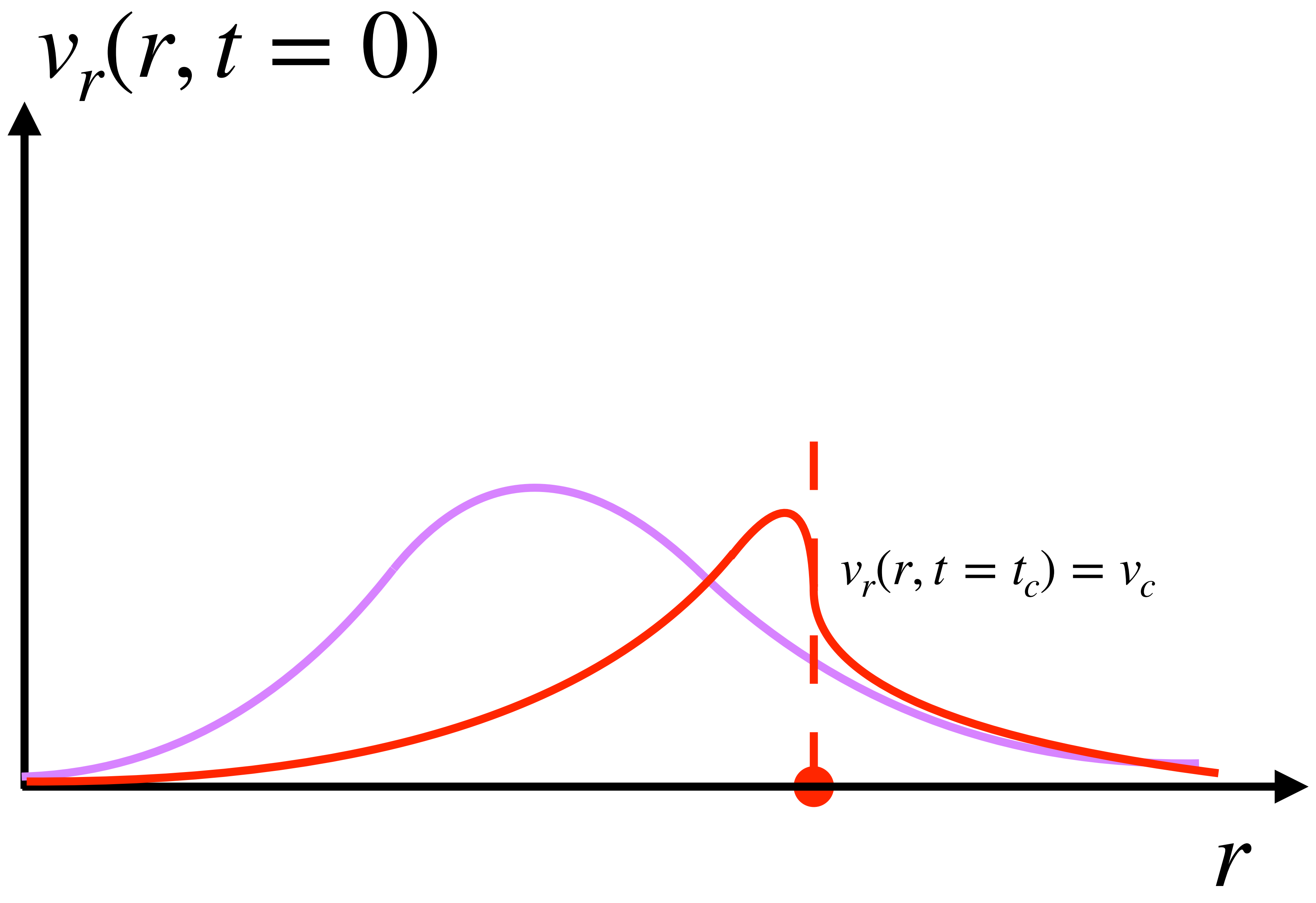
The search of singularities in fluid motion

- L. Lichentenstein (1925); N. Gunther (1927); J. Leray (1934).
- Onsager Conjecture (1949): The velocity field does not remain differentiable: $|\mathbf{v}(\mathbf{r}' - \mathbf{r}) - \mathbf{v}(\mathbf{r}')|^3 \sim \epsilon r$.
- A geometric approach: vortex sheets (Moore, 1979), vortex filaments (Siggia, 1985) [No intrinsic length].
- Numerical simulations: S. Orszag et al (1980-1982), Grauer & Sideris (1991), Pumir & Sigma (1992), Kerr (1993),… Gibbons (2007) summarizes: 9 Yes & 7 No.
- Luo & Hou (PNAS 2014) & Barkley (2020). Numerical evidence of a blow-up in **axisymmetric Euler** equation with a solid boundary.
- Theory: Pomeau (1995-2018), tried different scaling for Leray's self similar solutions.
- Math. point of view: Th. Beale, Kato, Majda (1984): There exists regular solutions of 3D Rule if $\int_0^T |\omega(t)|_\infty < \infty$. Necas, Ruzicka & Sverak (1996) no Leray self-similar solution for Navier Stokes, Chae (2010).
- Elgindi (Nov 2021) finds a finite-time singular solution of the **axisymmetric Euler** equations **without swirl,** but for a **singular** initial condition**.**
- Evidence of anisotropic finite-time singularity. (S. Rica, 2022)
- Physics Informed neural networks approach (Y. Wang, C-Y Lai, J.Gómez-Serrano, T. Buckmaster, PRL 2023).
- Chen & Hou (arxiv Oct2022) Proof of a blow-up in **axisymmetric Euler** equation **with a solid boundary**.

Geometry of the singularity: a rim-like singularity

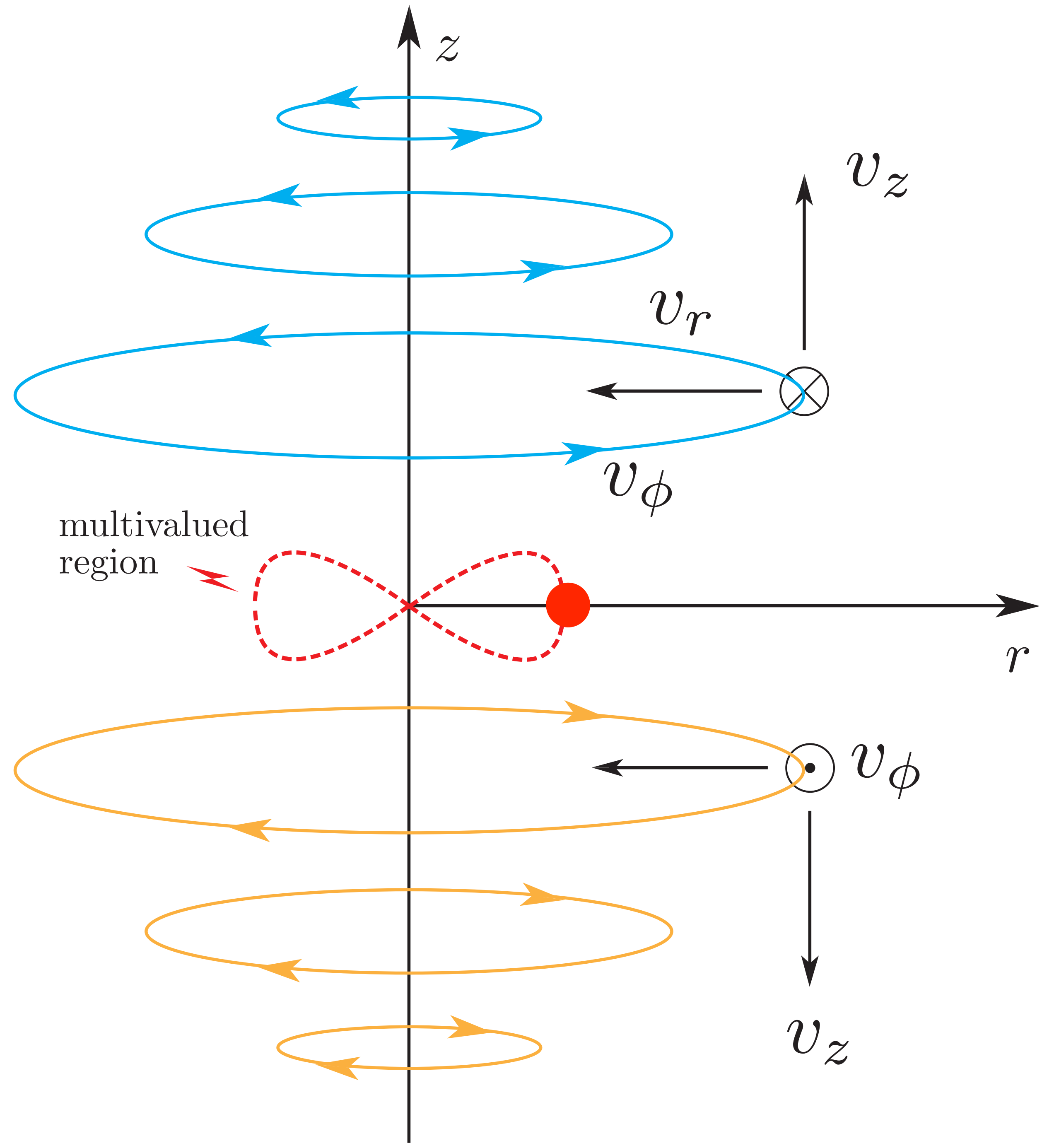
S. Rica, *"Potential anisotropic singularity …"* **PHYSICAL REVIEW FLUIDS 7, 034401 (2022)**

Assuming $\left|\frac{\partial\,\cdot}{\partial z}\right| \gg \left|\frac{\partial\,\cdot}{\partial r}\right|$ and using the Riemann's characteristic method one observes two possible types of singularities depending on the initial condition.



$$\frac{\partial v_r}{\partial r} \sim \frac{1}{(t_c - t)},$$

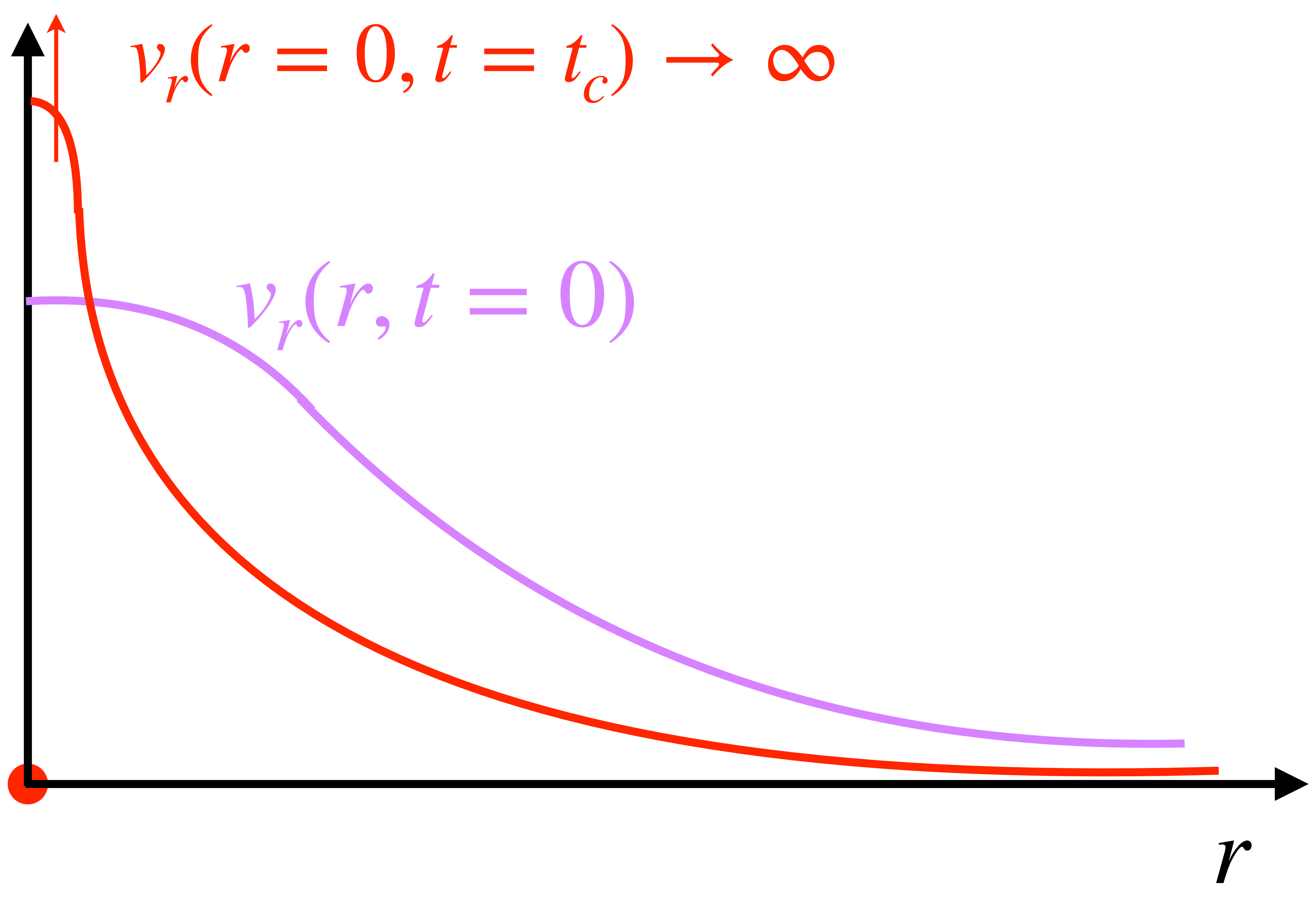
$$\frac{\partial v_\phi}{\partial r} \sim \frac{1}{(t_c - t)}$$

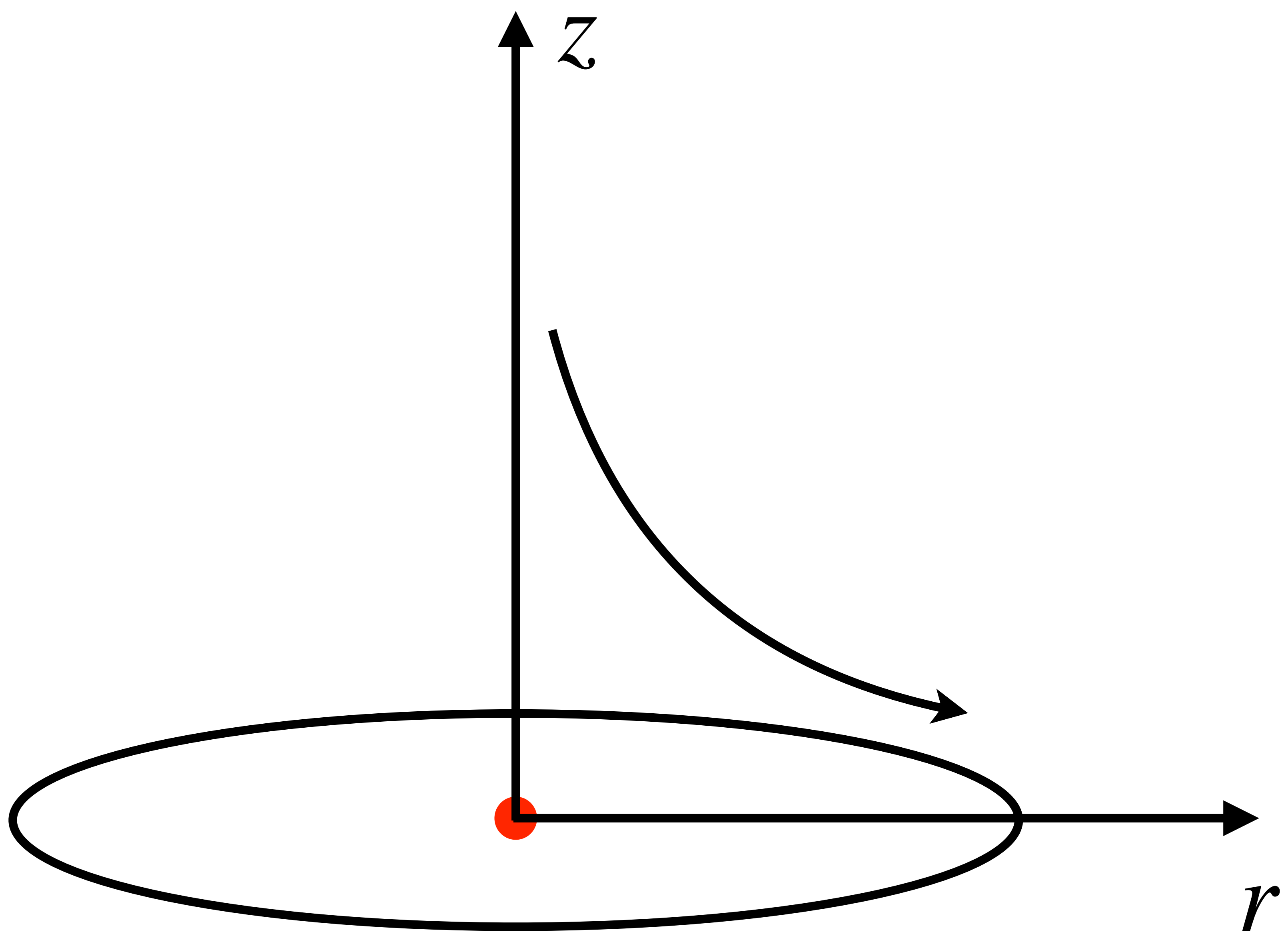


Geometry of the singularity: a point-like singularity

S. Rica, *"Potential anisotropic singularity …"* **PHYSICAL REVIEW FLUIDS 7, 034401 (2022)**

Assuming $\left|\frac{\partial\,\cdot}{\partial z}\right| \gg \left|\frac{\partial\,\cdot}{\partial r}\right|$





Axisymmetric flow

Discussion: The existence of singularities

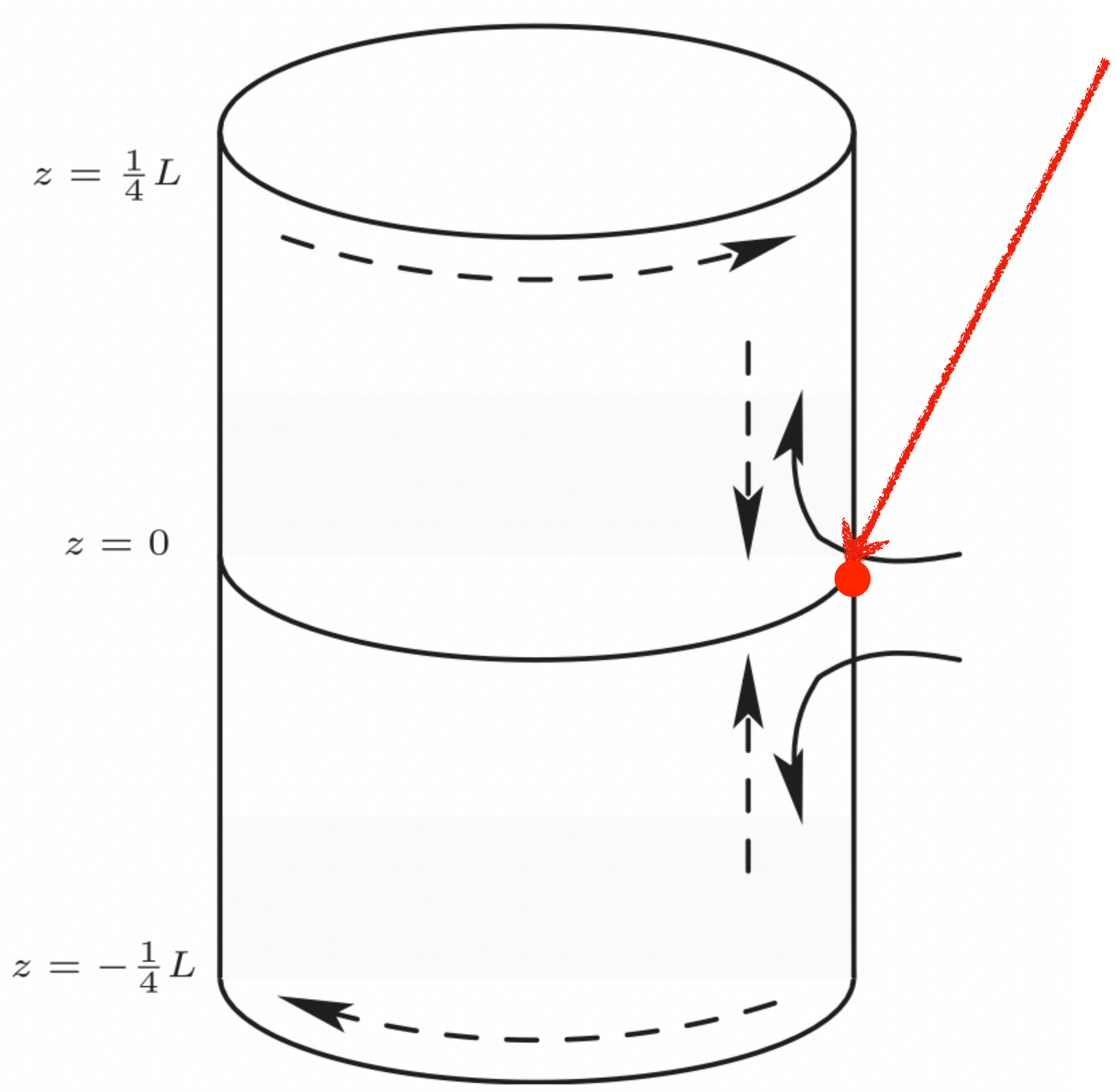
- may depend on the initial condition.
- may depend on the boundary conditions [Luo & Hou, PNAS, **111** (2014).]
- Luo & Hou observed a self similar singularity.

$$u_1(\tilde{x},t) \sim [t_s - t]^{\gamma_u} U\left(\frac{\tilde{x}-\tilde{x}_0}{[t_s-t]^{\gamma_l}}\right),$$

$$\boxed{\gamma_l \sim 2.91,} \; \gamma_u \sim 0.46,$$

- For the following we look for self similar singular solutions.

Singularity arises on a circular rim at the boundary



Pointlike singular solution

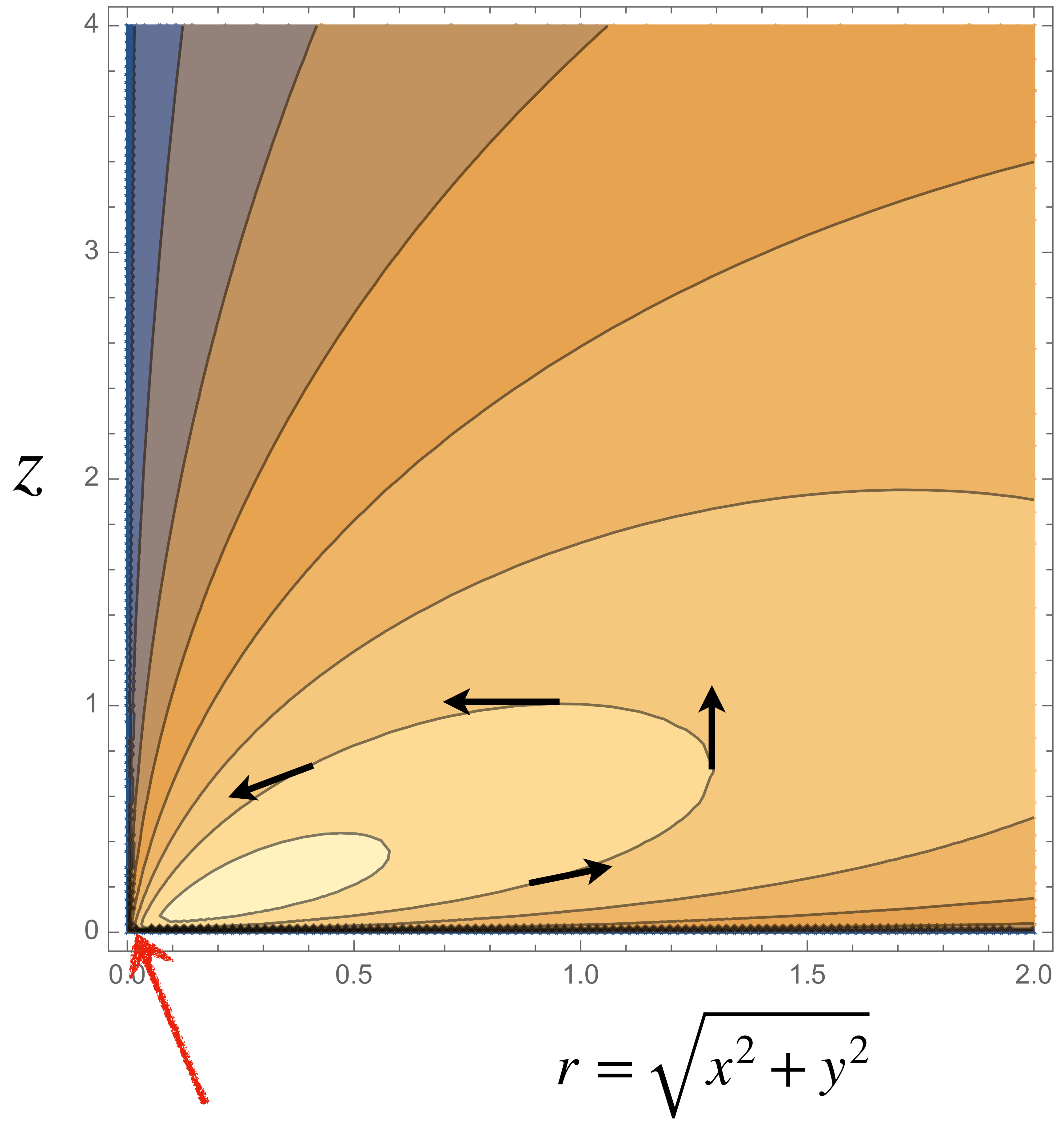
Elgindi (2021)

$v_\phi(x, y, z, t) = 0$ Zero swirl velocity.

$$\omega_\phi(x, y, z, t) = \frac{2\alpha}{c_*(\alpha)} \frac{\left((x^2 + y^2)\, z\right)^{\alpha/3}}{\left((t_* - t) + (x^2 + y^2 + z^2)^{\alpha/2}\right)^2}$$

NB. It is a family of solutions: $0 < \alpha < 1$. The transport term acting on ω_ϕ is negligible for $\alpha \ll 1$. But the initial condition is not smooth!!!

$\omega_\phi(r, z) = Cte$



Singular point

Self-similar singularities

- 1st kind

Explosion (Taylor - Sedov): $R(t) \sim \left(\frac{E}{\rho}\right)^{1/5} t^{2/5}$

$$v \sim r/tV\left(r/R(t)\right) \quad p \sim r^2/t^2P\left(r/R(t)\right)$$

Pinch-off of a Droplet (Eggers, Brenner, Lister, Stone ~1995)

$$v \sim \left(t_c - t\right)^{-1/2} V\left(x/\left(t_c - t\right)^{1/2}\right) \quad h \sim \left(t_c - t\right) H\left(x/\left(t_c - t\right)^{1/2}\right)$$

- 2nd kind:

Implosion (Sedov) $R(t) \sim (-t)^{\alpha}$

$$v \sim r/tV\left(r/R(t)\right) \quad p \sim r^2/t^2P\left(r/R(t)\right), \quad \alpha \approx 0.7$$

NLS focusing (Sulem et al)

$$\psi(r,t) = \frac{1}{(t_c - t)^{\frac{1}{2n}}}\phi\left(\frac{r}{\sqrt{t_c - t}}, -\log(t_c - t)\right)$$

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{1}{2}\nabla^2\psi - |\psi|^{2n}\psi \qquad \phi(\xi,\tau) = e^{i\lambda\tau}\varphi(\xi)$$

$\lambda \in \mathbb{R}$ is “nonlinear eigenvalue”

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The Euler equation for the velocity and the vorticity

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p$$

$$\nabla \cdot \mathbf{v} = 0$$

Define

$$\omega = \nabla \times \mathbf{v},$$

$$\frac{\partial \omega}{\partial t} + \nabla \times (\omega \times \mathbf{v}) = 0.$$

Self-similar singularities (Leray-Pomeau et al)

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p \qquad \nabla \cdot \mathbf{v} = 0$$

$$\mathbf{v}(\mathbf{r}, t) = \frac{1}{(t_c - t)^{1-\nu}} \mathbf{V}\left(\frac{\mathbf{r}}{(t_c - t)^{\nu}}\right)$$

$$p(\mathbf{r}, t) = \frac{\rho}{(t_c - t)^{2(1-\nu)}} P\left(\frac{\mathbf{r}}{(t_c - t)^{\nu}}\right)$$

$$(1-\nu)\mathbf{V} + \nu\left(\xi \cdot \nabla_\xi\right)\mathbf{V} + \left(\mathbf{V} \cdot \nabla_\xi\right)\mathbf{V} = -\nabla P$$

$$\nabla_\xi \cdot \mathbf{V} = 0$$

$\nu \in \mathbb{R}$ is “nonlinear eigenvalue”

Self-similar singularity for vorticity

$$\omega(\mathbf{r}, t) = \frac{1}{(t_c - t)} \boldsymbol{\Omega}\left(\frac{\mathbf{r}}{(t_c - t)^{\nu}}\right)$$

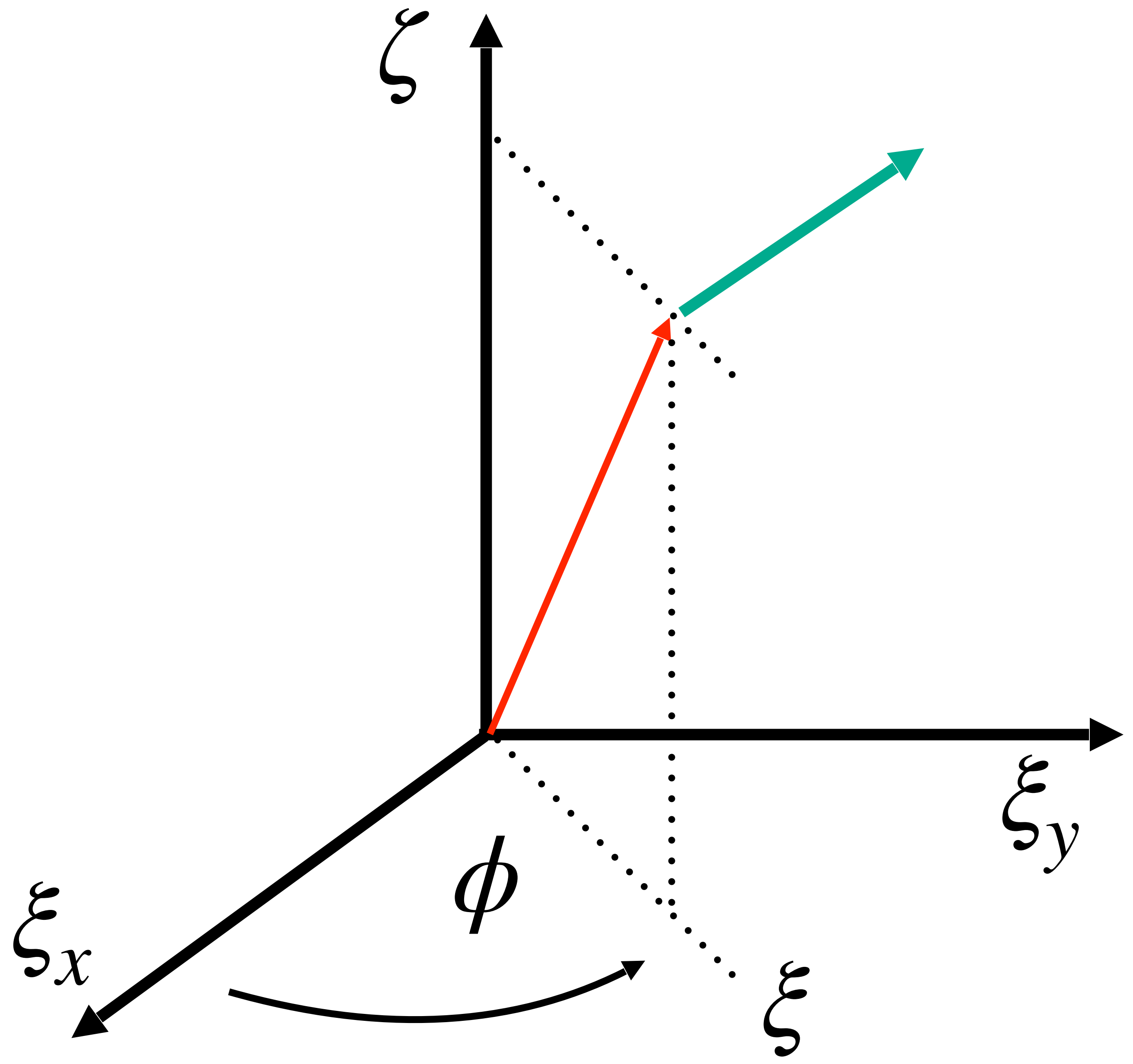
$$\boldsymbol{\Omega} + \nu\, \xi \cdot \nabla_\xi \boldsymbol{\Omega} + \nabla_\xi \times (\boldsymbol{\Omega} \times \mathbf{V}) = 0$$

$$\boldsymbol{\Omega} = \nabla_\xi \times \mathbf{V} \qquad \nabla_\xi \cdot \mathbf{V} = 0$$

$\nu \in \mathbb{R}$ is the same "nonlinear eigenvalue"

Axisymmetric Euler equations in cylindrical coordinates

$\mathbf{V}(\xi,\zeta) = V_\xi\,\hat{\xi} + V_\phi\,\hat{\phi} + V_\zeta\,\hat{\zeta}$ a 3D field



NB. 3 equations for 3 fields

$\Omega_\phi(0,\zeta) = V_\phi(0,\zeta) = \Psi(0,\zeta) = 0$

$$\Omega_\phi + \nu\left(\xi\partial_\xi + \zeta\partial_\zeta\right)\Omega_\phi + \frac{1}{\xi}\left(\partial_\xi\Psi\partial_\zeta\Omega_\phi - \partial_\zeta\Psi\partial_\xi\Omega_\phi\right) + \frac{1}{\xi^2}\partial_\zeta\Psi\Omega_\phi = -\frac{1}{\xi}\partial_\zeta V_\phi^2$$

$$(1-\nu)V_\phi + \nu\left(\xi\partial_\xi + \zeta\partial_\zeta\right)V_\phi + \frac{1}{\xi}\left(\partial_\xi\Psi\partial_\zeta V_\phi - \partial_\zeta\Psi\partial_\xi V_\phi\right) - \frac{1}{\xi^2}\partial_\zeta\Psi V_\phi = 0$$

$$\frac{1}{\xi}\left(\frac{\partial^2\Psi}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial\Psi}{\partial\xi} + \frac{\partial^2\Psi}{\partial\zeta^2}\right) = -\Omega_\phi.$$

∞ (in ζ) ∞ (in ξ)

$\Omega_\phi(\xi,0) = V_\phi(\xi,0) = \Psi(\xi,0) = 0$

Scheme of solution in steps

1. Expanding, $\Omega_\phi(\xi,\theta)$ and $V_\phi^2(\xi,\theta)$ in spherical hamonics using Legendre polynomials that satisfies the boundary conditions at $\theta = 0 \;\;\&\;\; \theta = \pi/2$.

2. To solve the stream function eq: $\frac{\partial^2\Psi}{\partial\xi^2} + \frac{1}{\xi^2}\left(\frac{\partial^2\Psi}{\partial\theta^2} - \cot\theta\frac{\partial^2\Psi}{\partial\theta}\right) = -\,\xi\sin\theta\,\Omega_\varphi(\xi,\theta)\,.$

3. To derive an infinite **autonomous** set of ode's for the amplitudes of each field, Ω_ϕ, V_ϕ^2 and Ψ . For instance $\Psi(\xi,\theta) = \xi^3\sum_{n=1}^{\infty}\left(N_n(\xi) + L_n(\xi)\right)y_n(\theta)$.

4. To close, the hierarchy and transform the problem into a $4N_*$ dimensional Dynamical System.

5. To solve the dynamical system by using a "shooting"-like method that determines shooting parameters, eg. the exponent ν.

6. To test the convergence.

Dynamical system approach

Finite $(4N_*)$ dynamical system

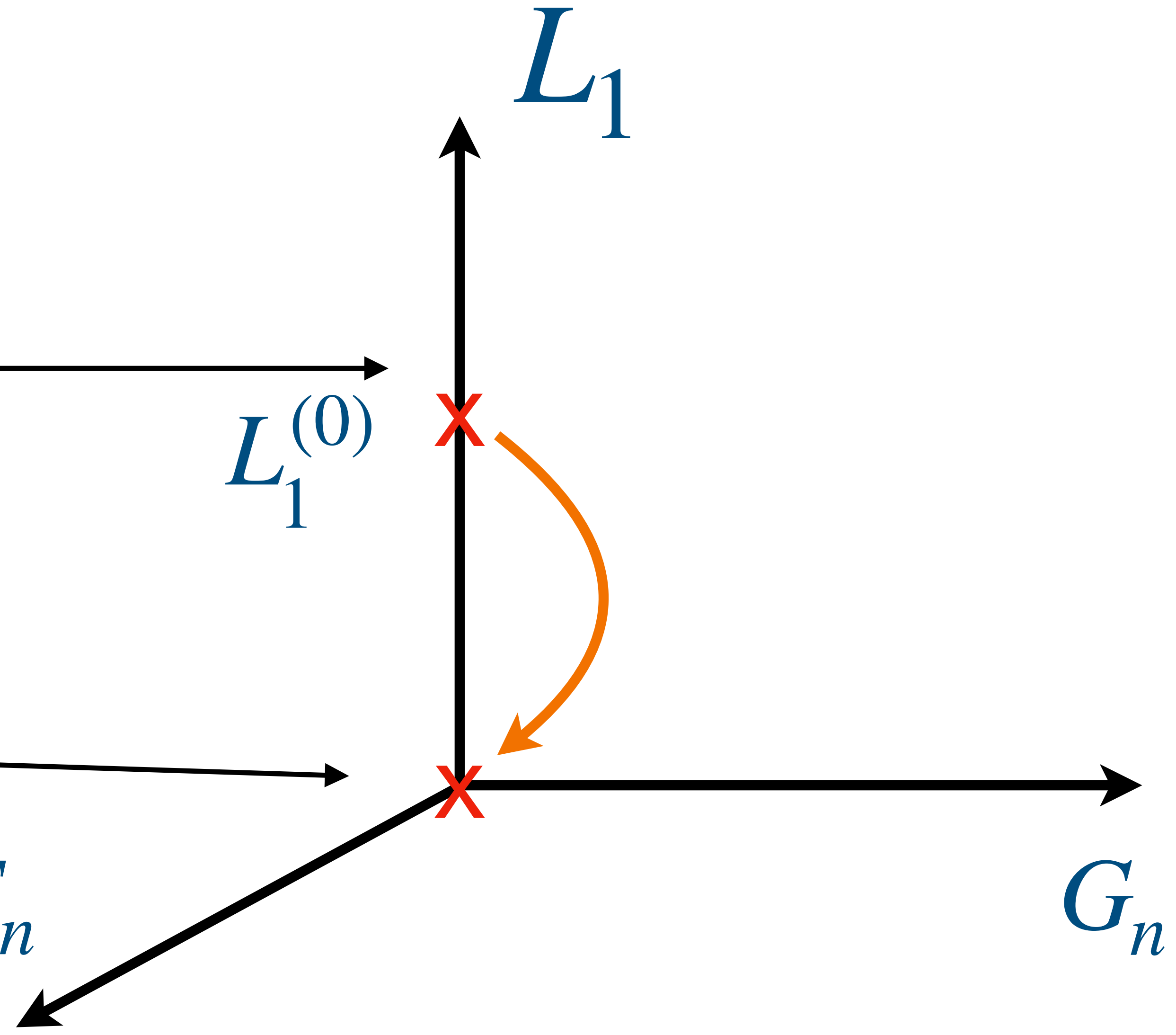
$4N_*$ ODEs   Heteroclinic trajectory among two fixed points.

The solutions are labeled by σ
that governs the inner behavior.
Smooth solutions require $\sigma = 1, 2, 3\ldots$

$\Omega_\phi(\xi, \theta) \sim \xi^\sigma \quad \text{for } \xi \to 0$

$$G_n \sim e^{\sigma s}\,,\ F_n \sim e^{\sigma s}\,,$$
$$N_n \sim e^{\sigma s}\,,\ L_{n>1} \sim e^{\sigma s}\,,\ L_1 - L_1^{(0)} \sim e^{\sigma s}$$
$$s \to -\infty$$

$F_n = G_n = N_n = L_{n>1} = 0, \&\, L_1 = L_1^{(0)}$



Stable fixed point

$F_n = G_n = N_n = L_n = 0$

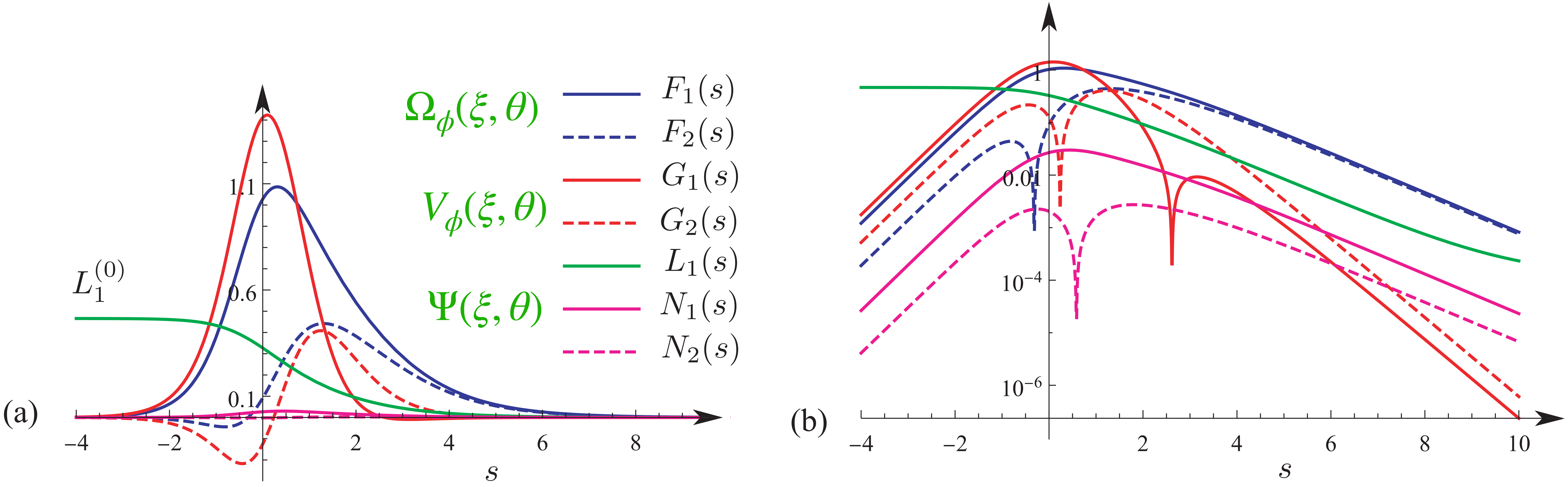
Selection problem:

$\nu \,\&\, L_1^{(0)}$

Numerical sketch: multi-parameter "shooting" method

Differential evolution (R. Storn and K. Price, *Journal of Global Optimization* **11**, 341, 1997)

Solutions for $N_* = 2$ and $\sigma = 2$

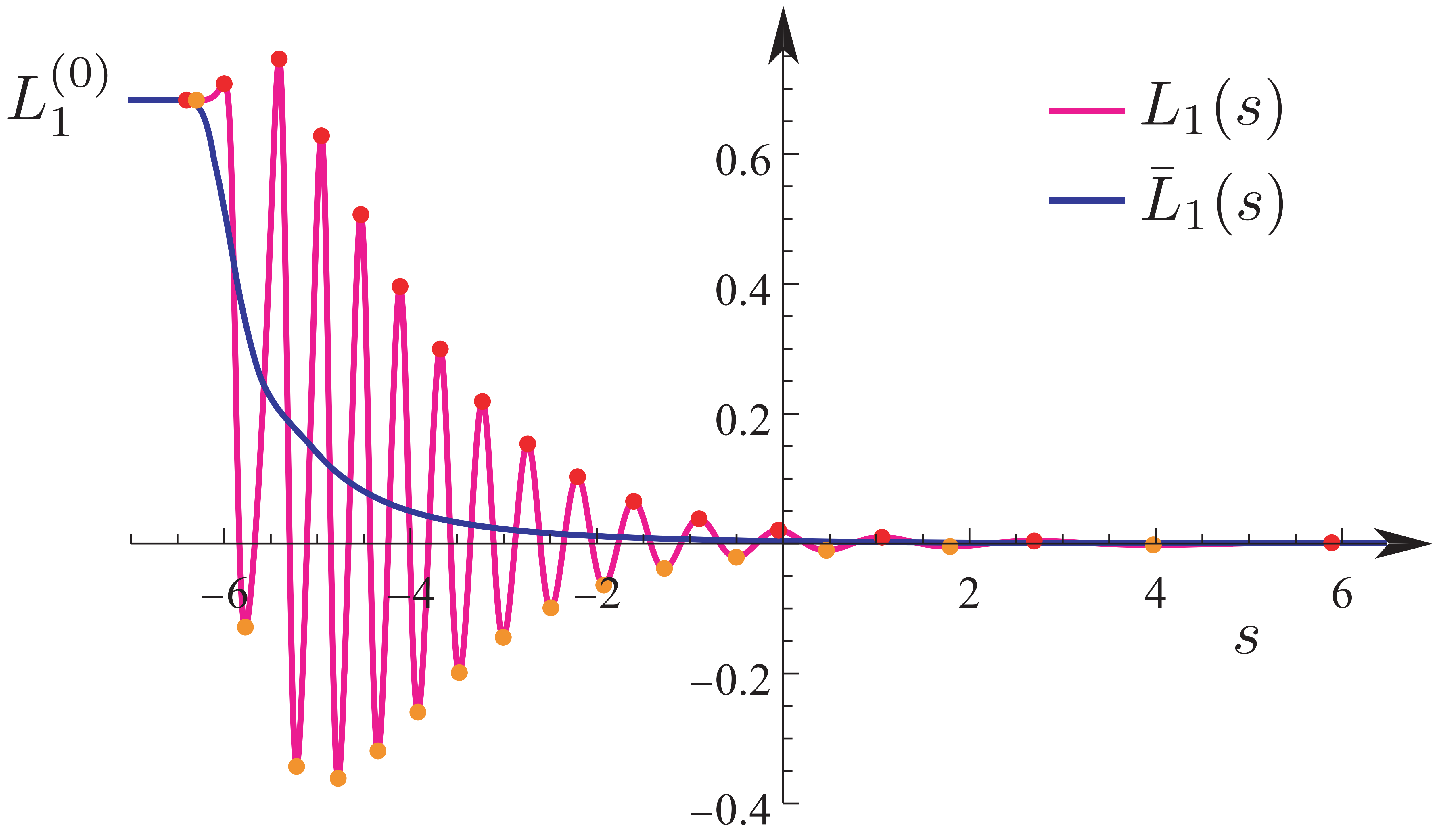


$\nu = 1.15231, \quad L_1^{(0)} = 0.46587.$

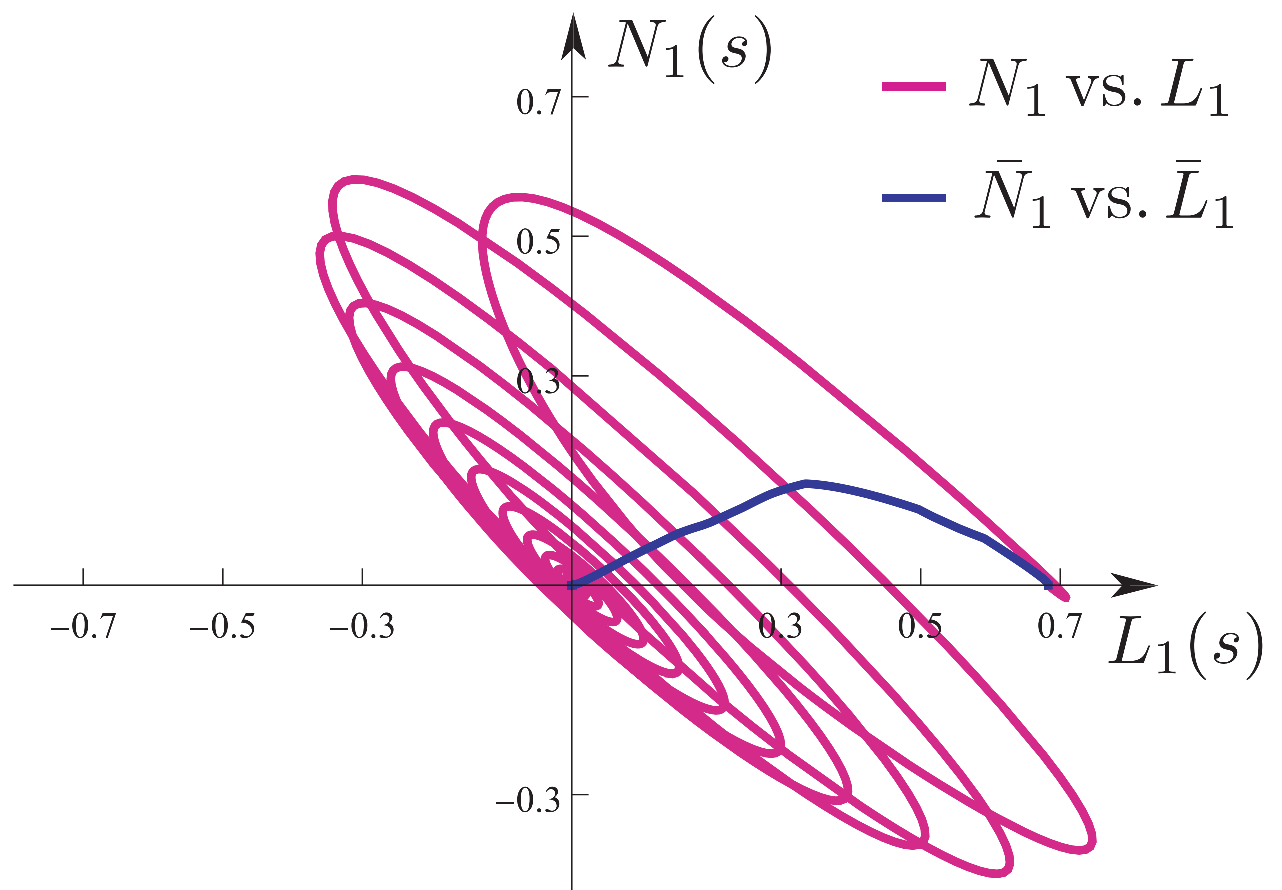
$\Omega_\phi(\xi,\theta) \sim \xi^\sigma \quad \text{for } \xi \to 0$

Numerical solutions for the truncated dynamical system

The solutions for different N^*

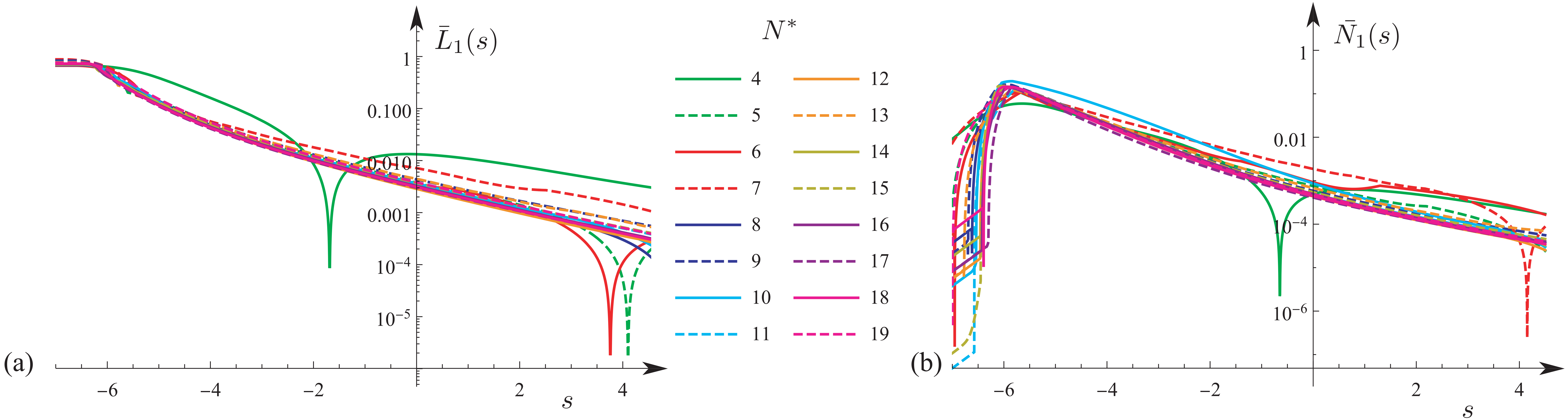


We use $L_1(s)$ for comparing different orders of truncation. Note that the horizontal axis is free thus one needs to fit a common origin. More relevant are the oscillations, probably due to the existence of oscillatory modes. Other amplitudes are less convergent.



Numerical solutions for the truncated dynamical system

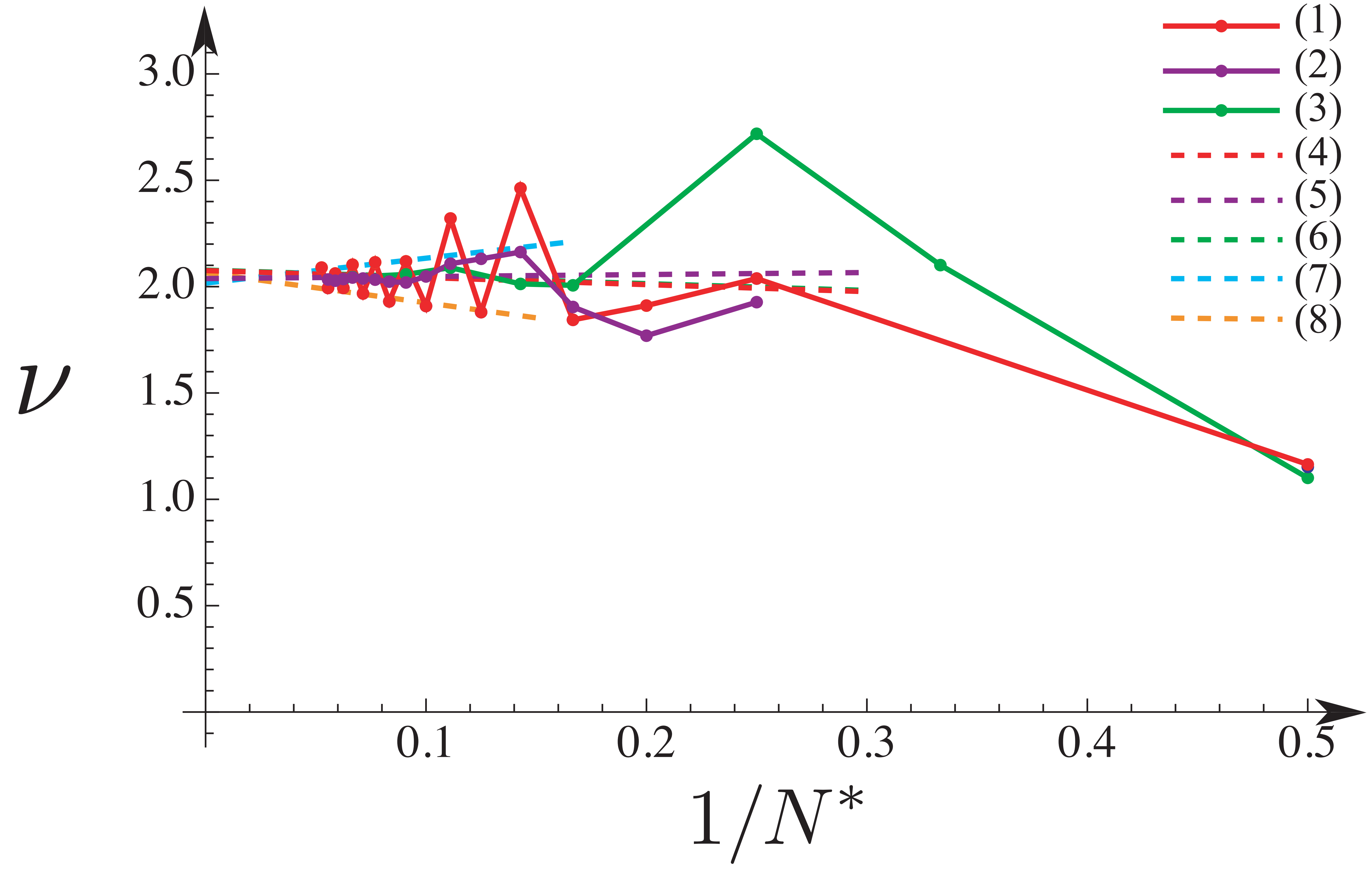
Removing the oscillations and setting a common origin for $L_1(s)$.



Numerical solutions for the truncated dynamical system

The solutions for different N

N^*	σ	# Nodes	ν_{N^*}	$S(\nu_{N^*})$
2	2	0	1.15231	-
3	2	-	-	-
4	2	1	2.03805	-
5	2	17	1.91086	1.76923
6	2	12	1.84385	1.90431
7	2	14	2.46257	2.1625
8	2	29	1.87994	2.13078
9	2	44	2.32043	2.10764
10	2	44	1.90880	2.04785
11	2	44	2.11878	2.01946
12	2	51	1.9303	2.02344
13	2	27	2.11442	2.03276
14	2	53	1.96769	2.03814
15	2	53	2.10321	2.04262
16	2	69	1.99363	2.0357
17	2	94	2.06191	2.02777
18	2	128	2.02397	2.03353
19	2	98	2.08961	-



$\nu \approx 2.03$

Summary

- The axisymmetric Euler equations are decomposed by a series expansion in a Legendre basis.
- The assumption of finite-time singularity maps the original nonlocal Euler equations into an infinite set of ordinary differential equations with a self-similar exponent as a nonlinear eigenvalue.
- The self similar exponent appears to converge to $\nu \approx 2$. However, the problem needs a better closure.
- The value $\nu = 2$, indicates $\mathbf{v}(\mathbf{x}, t) = g(t_c - t)\,\mathbf{V}\left(\mathbf{x}/(g(t_c - t)^2)\right)$ with g a quantity with dimensions of an acceleration, namely $g \sim \Gamma^8/E_0^3$. Thus if initially helicity is zero the singularity may be avoided.
- The effect of dissipation and regularity problem of Navier-Stokes equations.