Weak and strong wave turbulence in solvable models

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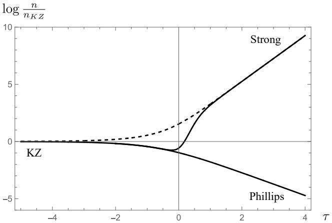
Waves

- Wave turbulence is sometimes weak; if the wind isn't too strong.
- But even if there is a range of scales in which it is weak, it (almost) always becomes strong, either at short or long scales. What happens there?
- [Newell Zakharov, '08]– dominated by cusp like objects (white caps). Generalized Phillips spectrum.
- So an option is to go from KZ $n_k \sim k^{-\gamma}$ at small k to $k^{-\widetilde{\gamma}}$ at large k.

Goal

- Construct models in which can study turbulence at strong nonlinearity. Observe transition from one scaling exponent to another as go from weak to strong nonlinearity



- Perturbatively [V.R. Smolkin, '22]

$$n_k = k^{-\gamma}\left(1 + \#\lambda k^{\frac{\beta}{3}-\alpha} + \dots\right)$$

Need to sum an infinite number of terms to get to strong nonlinearity.

Part I: Strongly local, large N theories

[V.R, Schubring, '24]

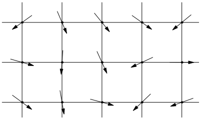
Two simplifying features.

1) A large number of fields ['t Hooft, '74] [Berges et al.]

Instead of one field, there are N fields a_p^i, $i = 1, \ldots, N$, grouped into a vector $\vec{a}_p$.

$$H = \sum_p \omega_p \, \vec{a}_p^{\,*} \cdot \vec{a}_p + \frac{1}{N} \sum_{p_1, \ldots, p_4} \lambda_{p_1 p_2 p_3 p_4} (\vec{a}_{p_1}^{\,*} \cdot \vec{a}_{p_3})(\vec{a}_{p_2}^{\,*} \cdot \vec{a}_{p_4}) \ .$$

λ is held finite, and can be small or large. N is taken to be very large, $N \gg 1$.



$O(N)$ spins, $N = 2$

2) A strongly local interaction

[Dyachenko, Newell, Pushkarev, Zakharov, '92]; [Grebenev, Medvedev, Nazarenko, Semisalov, '20]

$\lambda_{p_1 p_2 p_3 p_4}$ strongly peaked around momenta that are nearly equal $\vec{p}_1 \approx \vec{p}_2 \approx \vec{p}_3 \approx \vec{p}_4$.

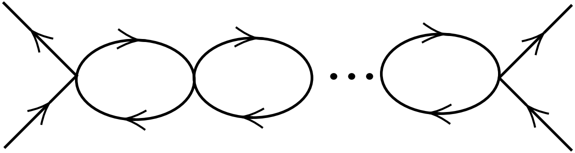
Standard kinetic equation

$$\frac{\partial n_k}{\partial t} = 4\pi \sum_{p_i} (\delta_{kp_1} + \delta_{kp_2} - \delta_{kp_3} - \delta_{kp_4}) |\lambda_{p_1 p_2 p_3 p_4}|^2 \prod_{i=1}^{4} n_i \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right) \delta(\omega_{12;34}) ,$$

becomes a differential equation

$$\omega^{\frac{d-\alpha}{\alpha}} \frac{\partial n}{\partial t} = \lambda^2 \frac{\partial^2}{\partial \omega^2} \left(\omega^{\frac{2\beta+3d}{\alpha}+2} n^4 \frac{\partial^2}{\partial \omega^2} \frac{1}{n} \right) ,$$

$$\omega_k \sim k^\alpha , \qquad \lambda_{p_1 p_2 p_3 p_4} \sim p^\beta .$$



Summing the geometric series of bubble diagrams gives the kinetic equation,

$$\omega^{\frac{d-\alpha}{\alpha}}\frac{\partial n}{\partial t} = \frac{1}{N}\frac{\partial^2}{\partial\omega^2}\left(\frac{\lambda^2\omega^{\frac{2\beta+3d}{\alpha}+2}n^4\frac{\partial^2}{\partial\omega^2}\frac{1}{n}}{\left|1-c\,\lambda\,\omega^{\frac{d+\beta}{\alpha}}\frac{\partial n}{\partial\omega}\right|^2}\right),$$

Stationary solutions:

$$\lambda^2\omega^2\frac{\partial^2}{\partial\omega^2}\frac{1}{n} = (P-Q\omega)\,\omega^{-\frac{3d+2\beta}{\alpha}}\left(n^{-2}-c\,\lambda\,\omega^{\frac{d+\beta}{\alpha}}\frac{\partial}{\partial\omega}\frac{1}{n}\right)^2 \quad .$$

Three asymptotic solutions

1) **Kolmogorov-Zakharov**: drop term proportional to λ:

$$\lambda^2 \omega^{\frac{2\beta+3d}{\alpha}+2} n^4 \frac{\partial^2}{\partial \omega^2} \frac{1}{n} = P \; , \quad \Rightarrow \quad n \sim \lambda^{-2/3} P^{1/3} k^{-d-\frac{2}{3}\beta}$$

2) **Strong turbulence**: drop the n^{-2} term,

$$\omega^2 \frac{\partial^2}{\partial \omega^2} \frac{1}{n} = P \omega^{-\frac{3d+2\beta}{\alpha}} \left(c\,\omega^{\frac{d+\beta}{\alpha}} \frac{\partial}{\partial \omega} \frac{1}{n} \right)^2 \; , \quad \Rightarrow \qquad n \sim c^2 P k^{-d-2\alpha}$$

3) **Phillips (Critical Balance)**: drop left side (large λ with $n \sim 1/\lambda$)

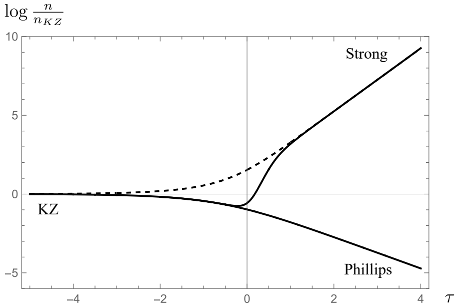
$$n^{-2} - c\,\lambda\,\omega^{\frac{d+\beta}{\alpha}} \frac{\partial}{\partial \omega} \frac{1}{n} = 0 \; , \quad \Rightarrow \qquad n \sim \frac{1}{c\,\lambda} k^{-(d+\beta-\alpha)}$$

Flowing between solutions

Strength of nonlinearity:

$$\epsilon_k = \frac{\lambda_{kkkk} n_k k^d}{\omega_k} \sim k^{\beta+d-\gamma-\alpha} \sim k^{\frac{\beta}{3}-\alpha} \ .$$

If $\beta > 3\alpha$: KZ at small k. At large k: strong turbulence or Phillips.



$\omega \sim e^{\tau}$

Comments

- Presented a model with a kinetic equation valid for both *weak* and *strong* interactions
- At strong nonlinearity, find a generalized Phillips spectrum (critical balance) and a new strong wave turbulence spectrum. Both of these can be obtained by “dimensional analysis”
- The challenge, however, is to have a consistent dynamical theory that achieves these scalings. That’s what our kinetic equation does.

Future

- Study time-dependent solution of our kinetic equation
- Study large N with realistic interaction. The strongly local interaction prevents any potential divergences, which can be physically relevant talk by Falkovich.
- How to understand the strong turbulence scaling? What quantity to compute?

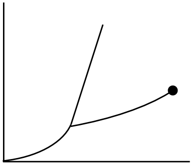
Part II: The ϵ expansion

V.R, M. Smolkin, in progress; [Gurarie, '95]

The previous discussion was a case in which we went to *strong* nonlinearity; the exponent changed by an *order* 1 amount.

Now: stay at *weak* nonlinearity, and get scale invariance (KZ scaling) at small/large scales and scale invariant at large/small scale with an exponent that differs by a *small* amount.

Wilson-Fisher '72: Describes water-vapor phase transition (a scale invariant state), by working in $4 - \epsilon$ dimensions.



Strength of nonlinearity: $\epsilon_k \sim \lambda k^{\frac{\beta}{3}-\alpha}$.

$$\omega_k \sim k^{\alpha}, \qquad \lambda_{p_1 p_2 p_3 p_4} \sim p^{\beta}.$$

$\beta = 3\alpha$ is special. If nonlinearity small at one scale, small at all scales.

We will take

$$\beta = 3\alpha(1-\epsilon) \ , \qquad \epsilon \ll 1$$

Nonlinearity grows slowly.

Is there a physical case when this is true? (or $\beta \approx 2\alpha$, for inverse cascade)

We will work at *small nonlinearity* (λ) and *small* ϵ.

A class of diagrams dominates, which we sum.

Result, for $\lambda_{1234} = \lambda_0 (p_1 p_2 p_3 p_4)^{\beta/4}$,

$$\frac{dn_1}{dt} = 16\pi \sum_{2,\ldots,4} \delta(\omega_{p_1 p_2; p_3 p_4}) \lambda_{1234}(\mu)^2 \prod_{i=1}^{4} n_i \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right)$$

where

$$\lambda_{1234}(\mu) = \frac{\lambda_0 (p_1 p_2 p_3 p_4)^{\beta/4}}{1 + \lambda_0 \frac{\mu^{-\epsilon}}{\epsilon}}$$

At small scales, recover KZ. At large scales, effective β that is $\beta + \epsilon$. Therefore,

$$n_k \sim k^{-d - \frac{2}{3}\beta - \frac{2}{3}\epsilon}$$

To clarify: normally, KZ is valid as long as $\epsilon_k \sim \lambda k^{\frac{\beta}{3}-\alpha}$ is small. One expects the corrections to scale as powers of ϵ_k,

$$n_k = k^{-\gamma} \left(1 + \#\epsilon_k + \ldots\right)$$

But, for small ϵ, there is an additional small parameter, and the expansion is

$$n_k = k^{-\gamma} \left(1 + \#\mathcal{L}_k + \ldots\right) \ , \quad \mathcal{L} \sim \epsilon_k/\epsilon$$

We work at small ϵ_k. KZ is for small $\mathcal{L}_k$, and new scaling is at large $\mathcal{L}_k$.

- For a general coupling, we believe the answer is the same for the exponent
- The exception is if there is an IR divergence in the loop integral
- The exponent in the IR can be found by dimensional analysis. Nontrivial thing is when actually get it. (In above example, need positive λ_0; defocusing).
- For a general coupling, form of coupling changes as flow from small scales to large scales. E.g.

$$\vec{p}_1 \cdot \vec{p}_3 \, \vec{p}_2 \cdot \vec{p}_4 \to \vec{p}_1 \times \vec{p}_3 \, \vec{p}_2 \times \vec{p}_4 + \cdots$$

- What other quantities can we compute that are richer than just n_k? (e.g. four-point correlator will be sensitive to form of λ_{1234}; not just scaling)