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The bound state attractor of strong optical wave turbulence

Dr Jonathan Skipp
Department of Applied Mathematics and Data Science

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HALT
Hydrodynamical Approach to Light Turbulence

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INSTITUT DE PHYSIQUE DE NICE

Sergey Nazarenko
Directeur de Recherche

Clément Colléaux
PhD Student

Aston University
BIRMINGHAM UK

Jason Laurie
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Context – coherent structures in wave turbulence

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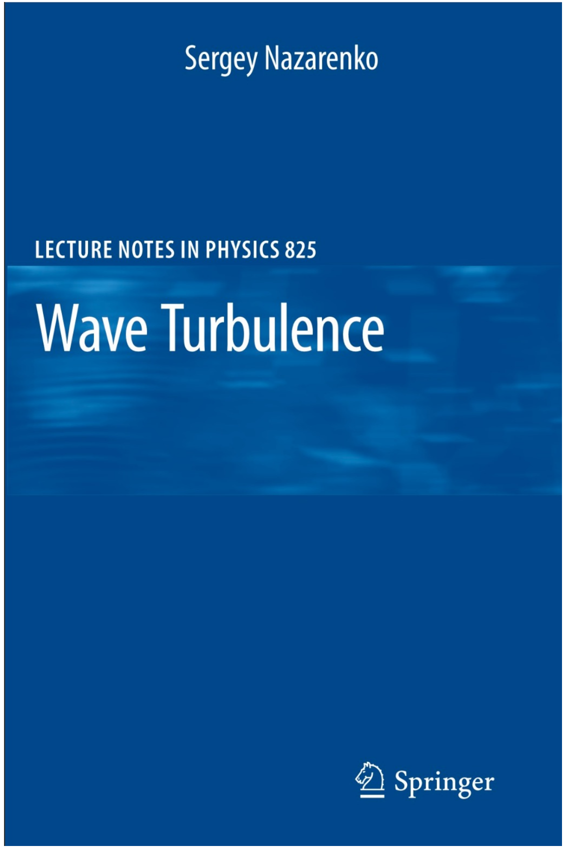

1.1 What is Wave Turbulence?

Wave turbulence (WT) can be generally defined as *out-of-equilibrium statistical mechanics of random nonlinear waves*. Often this definition is further narrowed to waves which are weakly nonlinear and dispersive, the cases when the mathematical description of WT is most systematic and unambiguous. However, we prefer to take an approach which focuses on WT as a general *physical phenomenon* without discarding a large number of systems which are commonly observed in nature but which have not been described rigorously yet. WT systems are

1.3.3 Discovery of Importance of Coherent Structures in WT Evolution

The most developed and systematic part of the WT theory deals with weakly nonlinear random waves. Some qualitative understanding has been recently achieved on the role of the coherent nonlinear structures randomly scattered through predominantly weak wave fields. The ultimate WT theory, yet to be developed in future, should include both components: incoherent weakly nonlinear waves and strongly nonlinear coherent structures. This theory should be able to describe the mechanisms of the WT cycle by which the weakly nonlinear random waves can get converted into the coherent structures and, conversely, breaking of the coherent structures with a partial return of energy to the incoherent waves. Obviously, such a theory could not be completely universal considering the fact that there is a great variety of the coherent structure types and the ways they can break depending on the particular physical system. However, some of the key

Here, we examine *solitons* ≈ coherent, localized, strongly nonlinear waves.

Single soliton: statistical attractor in nonintegrable systems

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Soliton turbulence

V. E. Zakharov, A. N. Pushkarev, V. F. Shvets, and V. V. Yan'kov
Scientific Council on Complex Problems in Cybernetics, Academy of Sciences of the USSR

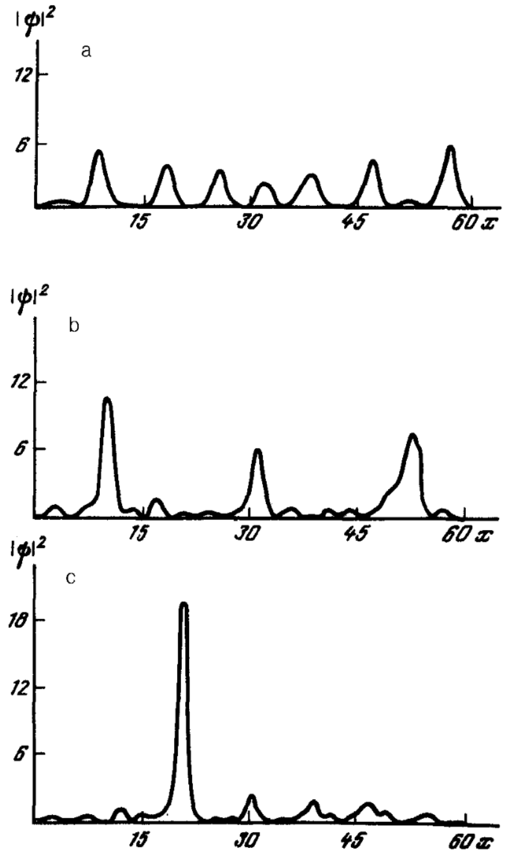
(Submitted 18 May 1988)
Pis'ma Zh. Eksp. Teor. Fiz. **48**, No. 2, 79–82 (25 July 1988)

A qualitative description of a strong wave turbulence in the absence of a wave collapse is given. The processes which lead to an increase in the amplitudes of the solitons as their number is reduced play the central role in the turbulence mechanism. In the conservative nonintegrable systems the soliton is a statistical attractor. The described picture is confirmed by a direct numerical simulation.

1. There is no doubt now that the development of a strong turbulence in various physical situations is accompanied by the formation of space-time structures which can be described in the coordinate space. The nonlinear Schrödinger equation is a sufficiently universal wave-turbulence model (see, e. g., Refs. 1–3):

$$i\psi_t + \Delta\psi + f(|\psi|^2)\psi = 0. \quad (1)$$

into solitons and slightly nonlinear free waves. The interaction of solitons with each other and with free waves accounts for the gradual transfer of waves from the weak solitons to stronger solitons. As a result, the amplitude of the solitons increases as their number decreases (Fig. 1b). At large time scales the system reaches a state in which it has a single soliton of small size and large amplitude (Fig. 1c). The measured velocity



Single soliton: statistical attractor in nonintegrable systems

PRL **95,** 263901 (2005) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2005

Condensation of Classical Nonlinear Waves

Colm Connaughton,[1] Christophe Josserand,[2] Antonio Picozzi,[3] Yves Pomeau,[1] and Sergio Rica[4,1]

this respect, an important insight was obtained from numerical simulations of *solitons* in the focusing, nonintegrable NLS equation [2]. These studies revealed that the nonlinear wave would generally evolve to a state containing a large-scale coherent localized structure, i.e., solitary-wave, immersed in a sea of small-scale turbulent fluctuations. The solitary wave is a "statistical attractor" for the system, while the fluctuations contain, in principle, all information necessary for time reversal. Importantly, the solitary-wave solution minimizes the energy (Hamiltonian), so the system actually relaxes towards the state of lowest energy [2]. Only recently has a statistical description of this self-organization been developed [3,4]. Remarkably, when such systems are constrained by an additional invariant (e.g., the mass), the increase in entropy of small-scale turbulent fluctuations *requires* the formation of coherent structures to "store" this invariant [4], so that it is thermodynamically advantageous for the system to approach the ground state which minimizes the energy [2].

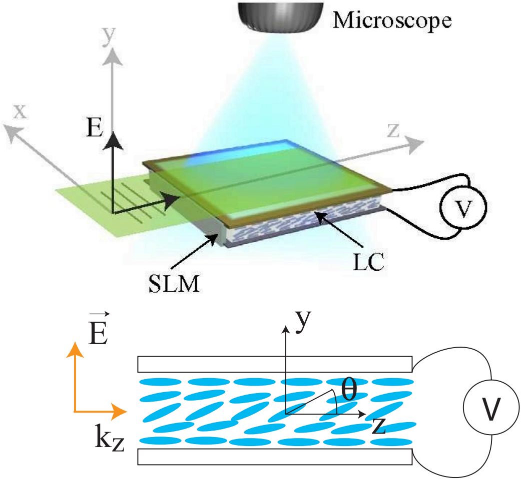
[2] V. E. Zakharov *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **48**, 79 (1988) [JETP Lett. **48**, 83 (1988)]; S. Dyachenko *et al.*, Zh. Eksp. Teor. Fiz. **96**, 2026 (1989) [Sov. Phys. JETP **69**, 1144 (1989)].
[3] R. Jordan, B. Turkington, and C. L. Zirbel, Physica (Amsterdam) **137D**, 353 (2000); R. Jordan and C. Josserand, Phys. Rev. E **61**, 1527 (2000).
[4] B. Rumpf and A. C. Newell, Phys. Rev. Lett. **87**, 054102 (2001); Physica (Amsterdam) **184D**, 162 (2003).

Optical wave turbulence (experiment)

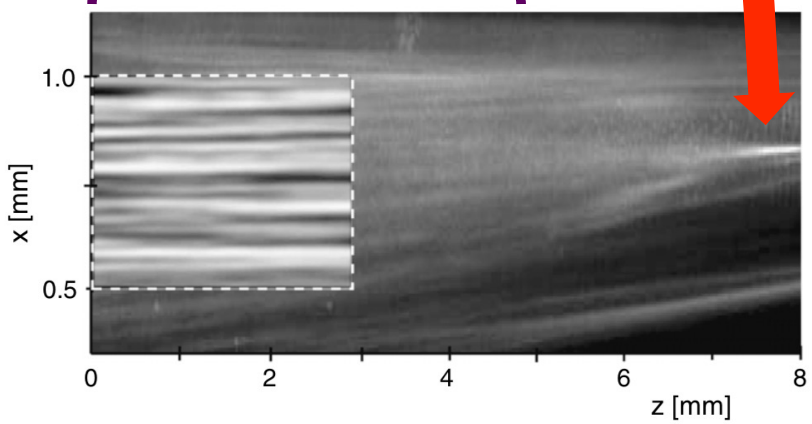
2280 J. Opt. Soc. Am. B/Vol. 26, No. 12/December 2009

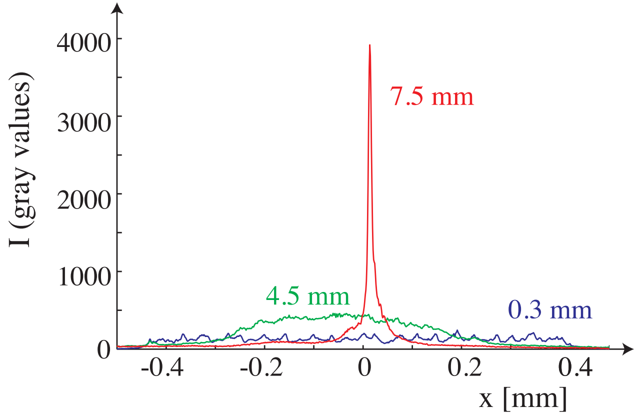
Optical wave turbulence and the condensation of light

Umberto Bortolozzo,[1] Jason Laurie,[2,*] Sergey Nazarenko,[2] and Stefania Residori[1]



Wave turbulence **Soliton formation**





Optical wave turbulence (theory)

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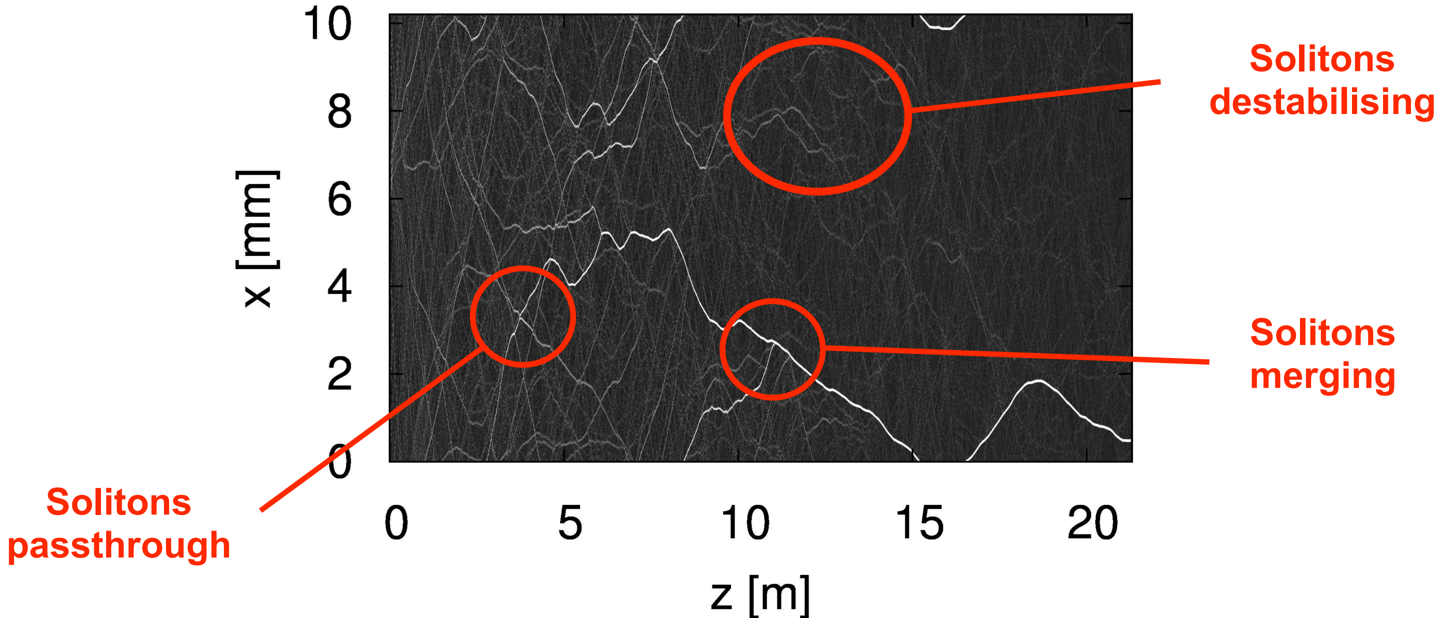
One-dimensional optical wave turbulence: Experiment and theory

Jason Laurie [a,*], Umberto Bortolozzo [b], Sergey Nazarenko [c], Stefania Residori [b]

Physics Reports 514 (2012) 121–175

$$2iq\frac{\partial\psi}{\partial z} = -\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2l_\xi^2\tilde{\psi}^2}\left(\psi|\psi|^2 + l_\xi^2\psi\frac{\partial^2|\psi|^2}{\partial x^2}\right)$$



Goal of this work

Goal of this work

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Here, we aim to characterize the "statistical attractor" of nonintegrable turbulence, and identify processes involved in its formation and consolidation.

We take as our working model the ***Schrödinger-Helmholtz equation (SHE)*** for the propagation of light through nonlinear photorefractive crystals.

Beam envelope

Change in refractive index

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu = 0$$

Beam intensity

Axial distance

$$\left(1 - \beta\frac{\partial^2}{\partial x^2}\right)V = |u|^2$$

Kerr coefficient

Nonlocality parameter

Weak wave turbulence → reduction to a ***semilocal approximation model,***
see JS, Laurie, Nazarenko (2023), Colléaux, JS, Laurie, Nazarenko (in prep)

Methods

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- Pseudo-spectral simulations of the SHE.
  - Periodic boundary conditions.
  - No forcing/dissipation.
  - ETDRK4 timestepping, 3/2-rule antialiasing.
  - We monitor the total energy H and total waveaction N (dynamical invariants), and waveaction spectrum near k_{max} to ensure good convergence.
  - Vary initial conditions to investigate different mechanisms.

$$H = \frac{1}{2}\int\left(|\partial_x u|^2 - \left[\left(1-\beta\partial_{xx}^2\right)^{-1/2}|u|^2\right]^2\right)dx$$

$$N = \int |u|^2\, dx$$

- Two main diagnostics:
  - Direct Scattering Transform of the cubic nonlinear Schrödinger equation.
  - Spatiotemporal (k, ω) spectrum.

Nonlinear Schrödinger Equation and its soliton

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The ***Nonlinear Schrödinger Equation (NLSE)*** is the leading-order equation that models the envelope of quasi-monochromatic light in the paraxial approximation, in a media with a local nonlinearity (the Kerr effect):

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0.$$

The NLSE has a single-soliton solution in infinite space,

$$u(x,t) = A\,\mathrm{sech}[A(x-vt)]e^{ivx}e^{i(A^2-v^2)t/2}e^{i\phi},$$

which exactly balances self-focusing with dispersion. It travels with constant speed v, without altering its amplitude A or sech-profile (ϕ is its initial phase).

Two solitons pass through each other ***elastically***: their outgoing v_j, A_j equal their incoming values, and they only undergo a shift in phase.

Direct Scattering Transform for the NLSE

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The NLSE is ***integrable***, in the sense of exact solubility via the ***Direct Scattering Transform (DST),*** a.k.a. Nonlinear Fourier Transform and associated Inverse Scattering Transform (Zakharov & Shabat 1972). Schematically:

Associated spectral problem

$$u(x,t) \longrightarrow \begin{pmatrix} \partial_x & -u \\ -u^* & -\partial_x \end{pmatrix} \Phi = -i\zeta\Phi \longrightarrow$$

Discrete eigenvalue spectrum $\{\zeta_j\}$, norming constants $r(\zeta_j)$, reflection coefficients $\rho(\xi)$

The eigenvalues $\zeta_j \in \mathbb{C}$ are in 1-to1 correspondence with the solitons in the system, and are constant ***(isospectrality)***.

Definition "what is a soliton?" → DST eigenvalue!

For NLSE, $\zeta_j^{\mathrm{Re}} = -v_j/2$ and $\zeta_j^{\mathrm{Im}} = A_j/2$ ***(e'value-amplitude-velocity relations)***.

DST as a diagnostic in the SHE

We explore the utility of applying the DST as a ***diagnostic tool*** to detect solitons in the SHE (Turitsyn+ 2019, Sugavanam+ 2017).

We study the SHE in a ***"nearly integrable" regime***: $\beta = 0.01 \ll 1$.

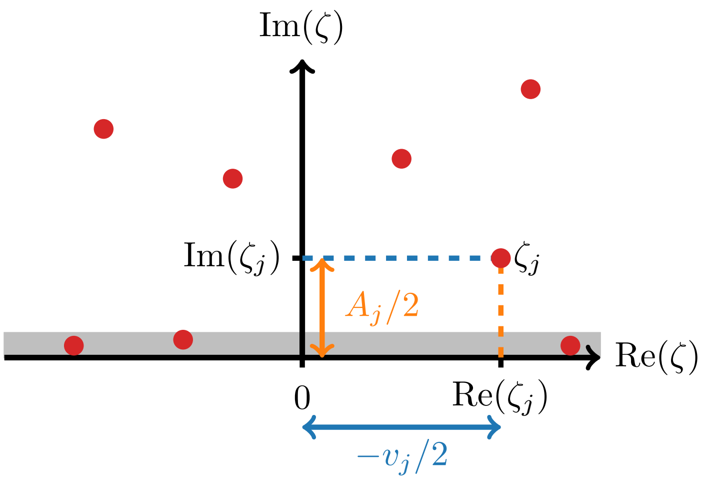
Deviation from integrability means we should not expect isospectrality, nor the eigenvalue-amplitude-velocity relations.

We use a Fourier colocation method to calculate the DST spectrum (Yang 2010).

The method creates spurious eigenvalues on the real axis.

We define a threshold $\zeta_{th}^{Im} \sim$ amplitude of an NLSE soli with FWHM $= L/4$.

(a) Direct Scattering Transform Eigenvalues



(k, ω) spectrum

Spatiotemporal Fourier transform

Spatiotemporal spectrum

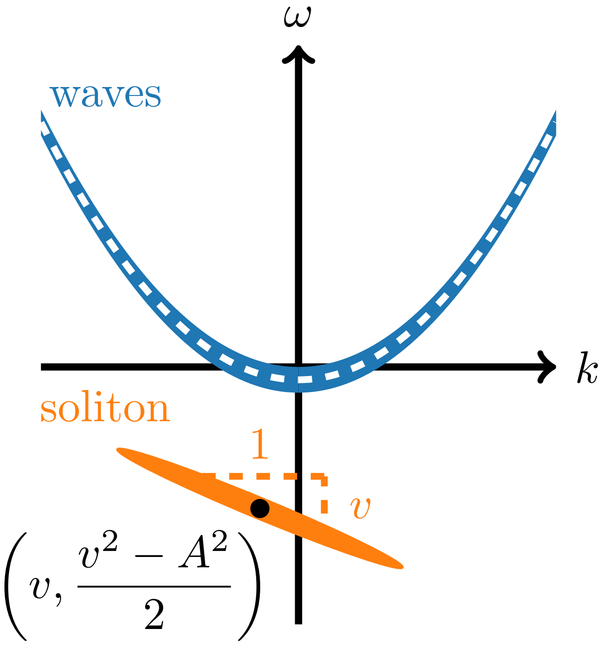
$$u(x,t) \longrightarrow \hat{u}(k,\omega) \longrightarrow |\hat{u}(k,\omega)|^2$$

Weak waves appear as a parabola:
dispersion relation $\omega_k = k^2/2$.

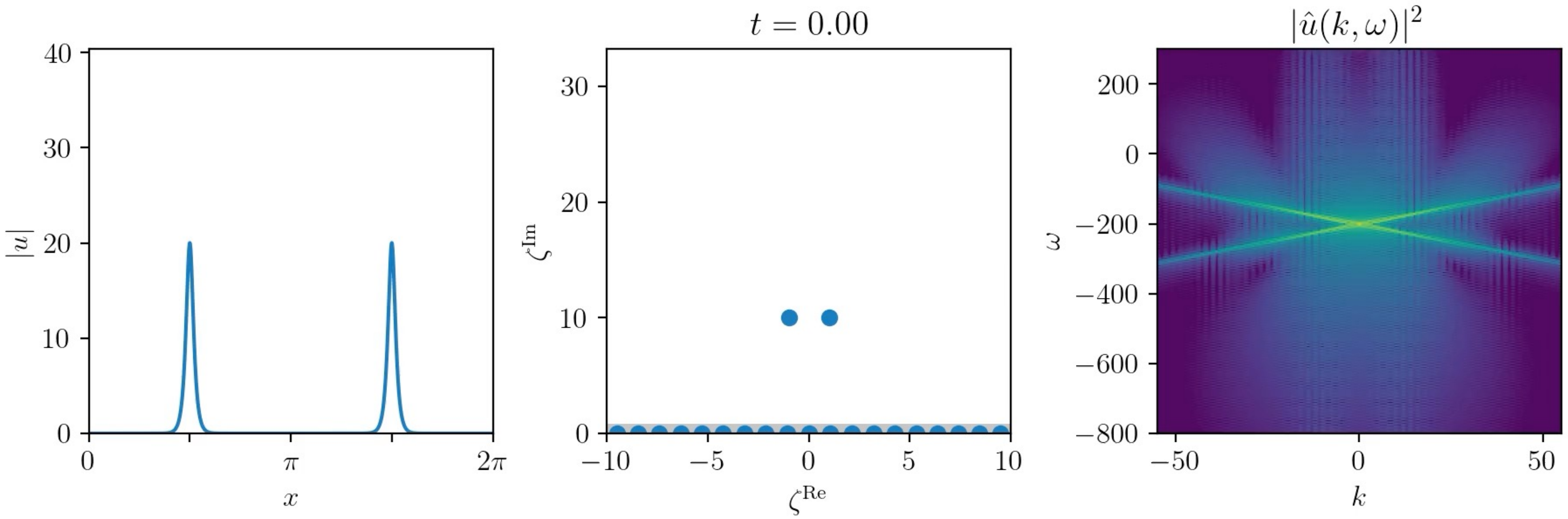
Downwards displacement of the parabola – nonlinear frequency correction.

Solitons appear as linear traces.

(b) Spatiotemporal (k, ω) Spectrum



Example: two solitons colliding in the NLSE



Results

IC 1: soliton turbulence leading to bound state

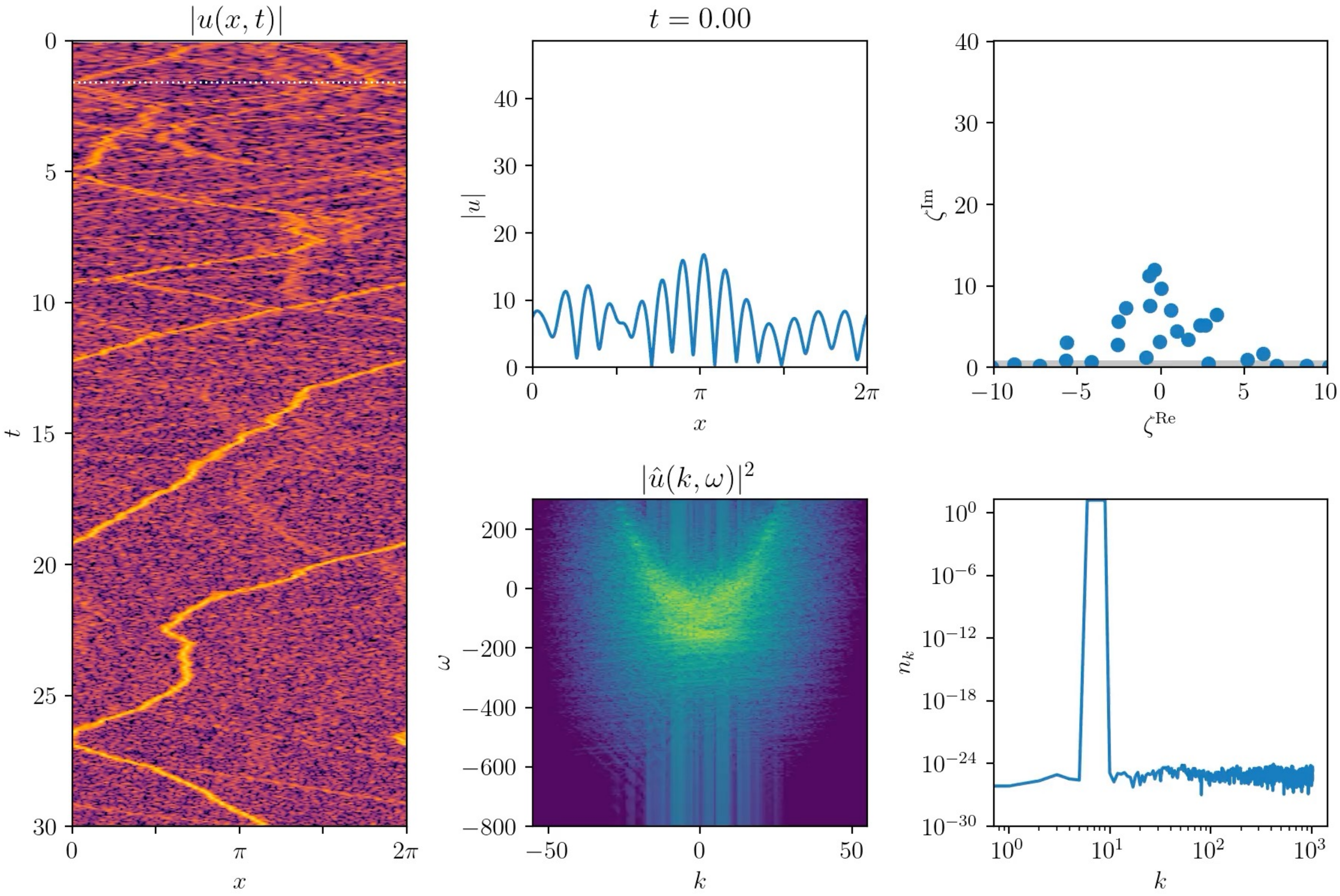
Initialise with large-scale random waves leading to soliton turbulence.

$N = 400, \;\; k_l = 6, k_u = 9.$

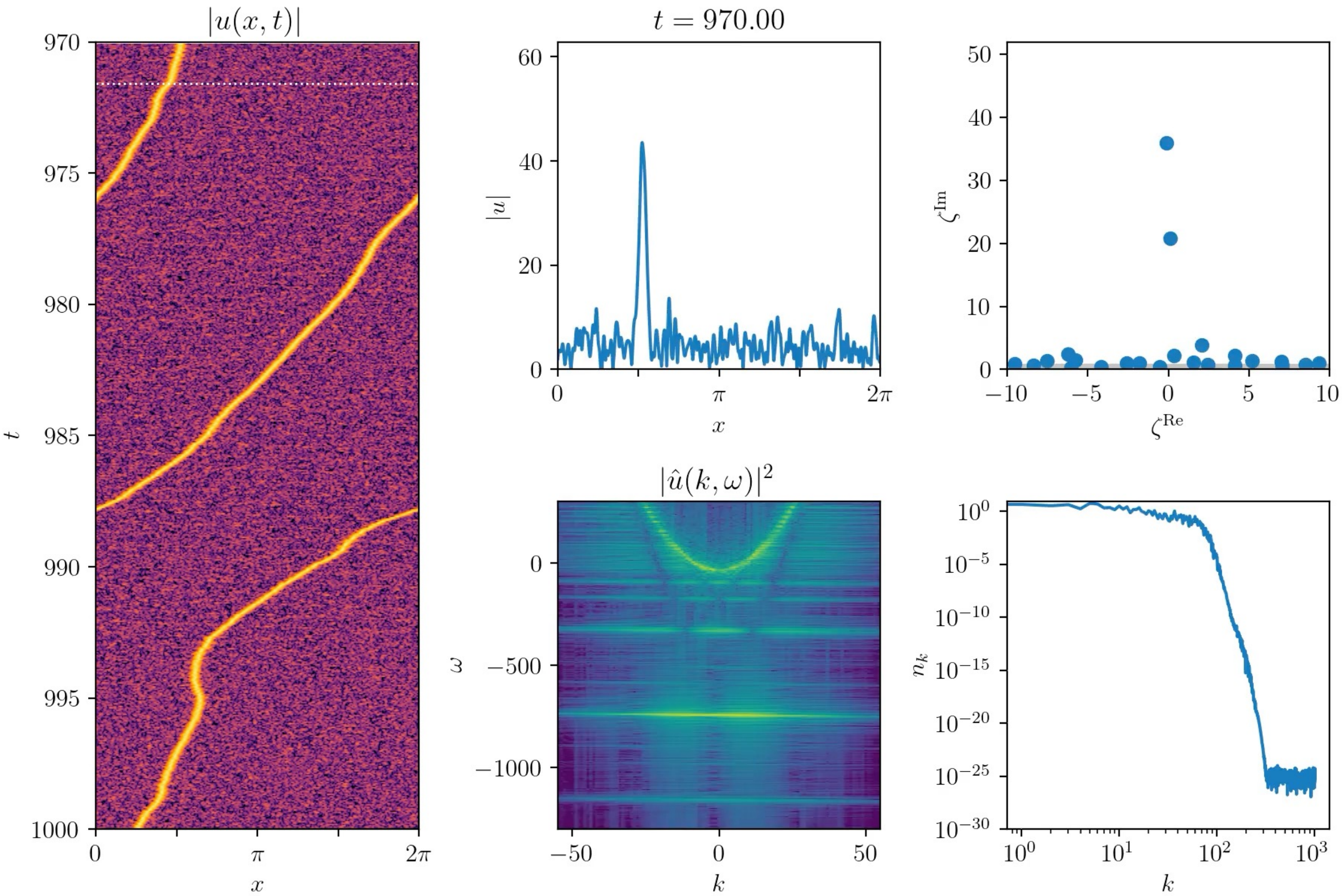
$$\hat{u}_k(0) = \begin{cases} Ae^{i\theta_k} & k_l \leq |k| \leq k_u, \\ 0 & \text{otw.} \end{cases} \qquad A = N/2L(k_u - k_l + 1), \quad \theta_k \sim \mathcal{U}[0, 2\pi).$$

- Coherent, interacting structures emerge (eigenvalues swarm in the complex plane).
- These ***collide and merge,*** until there is one dominant coherent solitary wave remaining. This dominant solitary wave oscillates in height/width.
- The DST spectrum has ***two dominant eigenvalues,*** ζ_1 and ζ_2, that oscillate out of phase. ζ_1 oscillates in phase with $\max_x |u(x,t)|$.
- ***Solitonic traces peel off the dispersion relation*** and move downwards in (k, ω) spectrum. The final dominant structure has ***one primary and two secondary traces.***

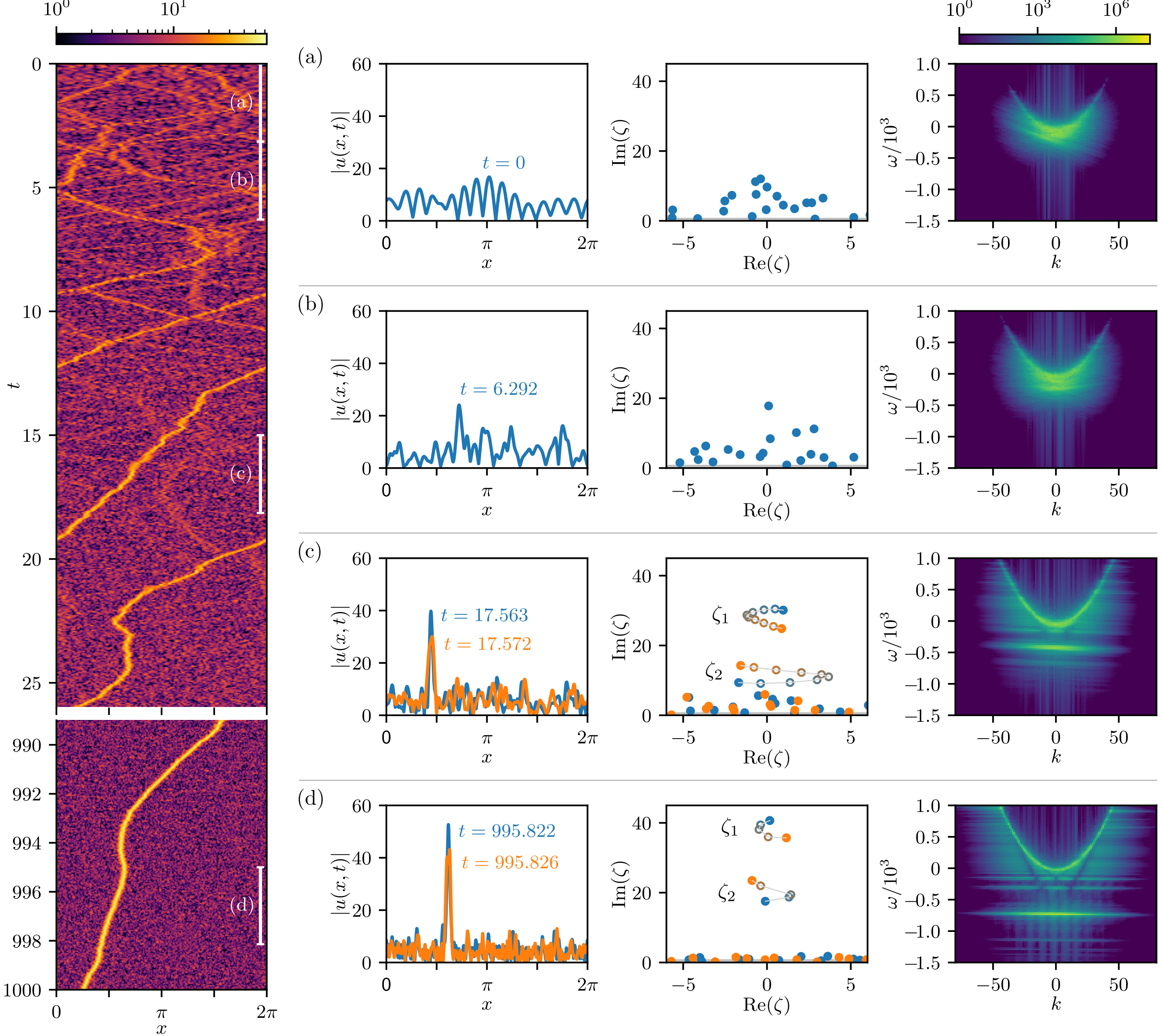
Early times



Late times



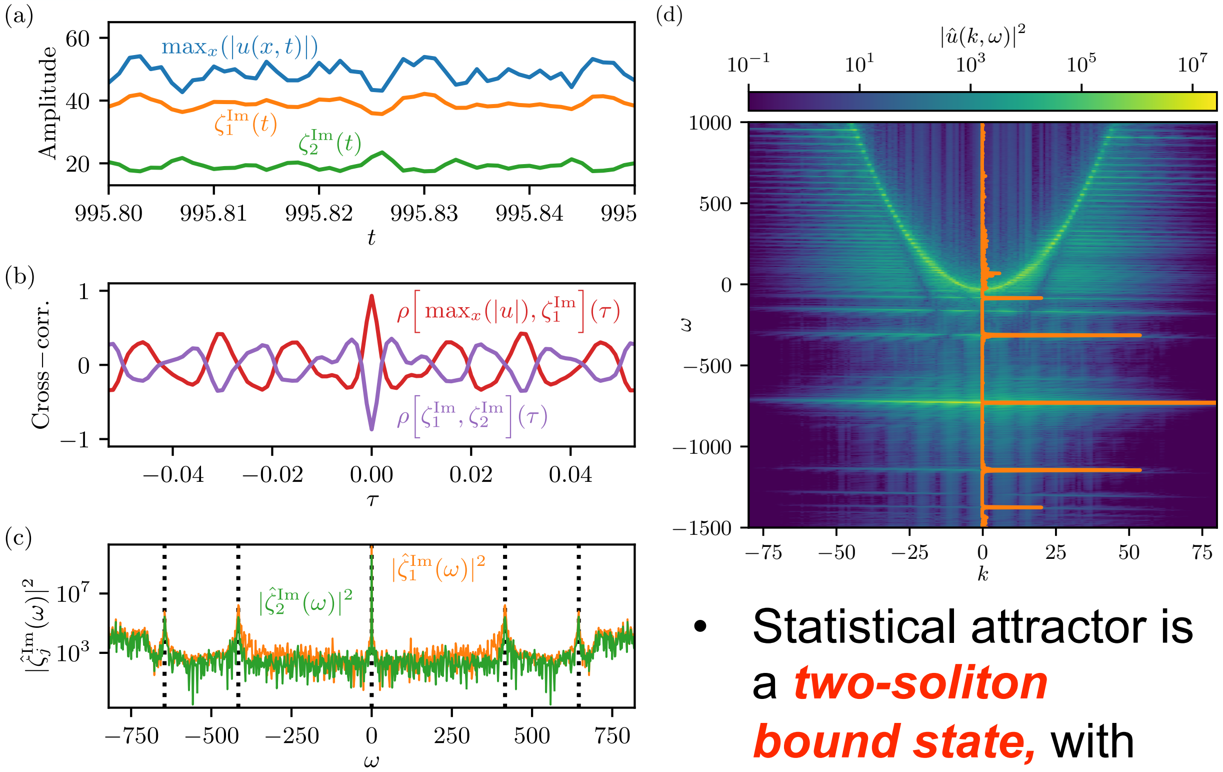
$|u(x,t)|$
$t = 970.00$
$|\hat{u}(k,\omega)|^2$
t
x
$|u|$
ζ^{Im}
ζ^{Re}
ω
k
n_k



(a)
(b)
(c)
(d)
t
x
$|u(x,t)|$
$t = 0$
$t = 6.292$
$t = 17.563$
$t = 17.572$
$t = 995.822$
$t = 995.826$
$\mathrm{Im}(\zeta)$
$\mathrm{Re}(\zeta)$
ζ_1
ζ_2
$\omega/10^3$
k

The "statistical attractor" is a bound state

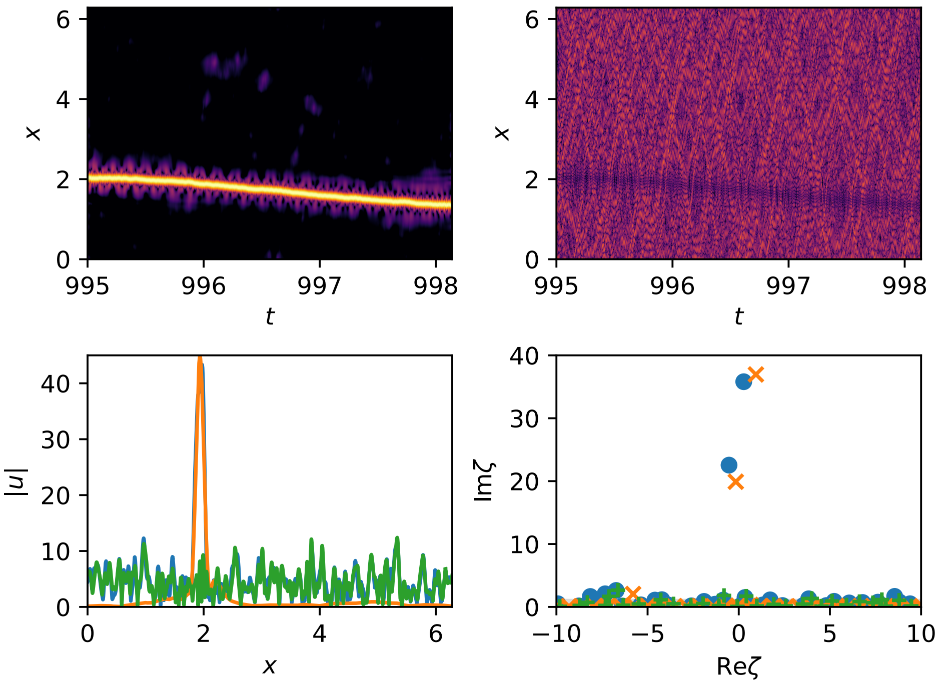
- Isospectrality of the eigenvalues is broken.
- $\bar{A}/\zeta_1^{\mathrm{Im}} = 1.25$, $\bar{v}/\zeta_1^{\mathrm{Re}} = -2.12$
- ζ_1 and ζ_2, and $\max_x(|u|)$ all oscillate with definite phase relationships.
- The peaks of the PSD of these timeseries map onto the primary and secondary traces in the (k, ω) spectrum.
- This associates the dominant coherent wave with these two eigenvalues.

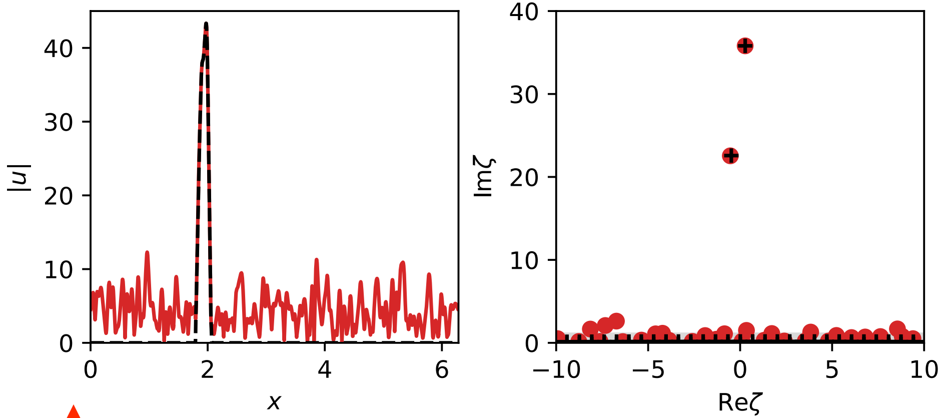


- Statistical attractor is a ***two-soliton bound state,*** with one soliton trapped inside the other.

The "statistical attractor" is a bound state

Further evidence to support this conclusion:





We can cut the dominant solitonic wave out of the $u(x,t)$ field and then take the DST. ζ_1 and ζ_2 reproduced exactly.

Likewise, we can cut out portions of the (k,ω) spectrum and reconstruct components. The dominant structure is reconstructed by the portion below the dispersion relation.

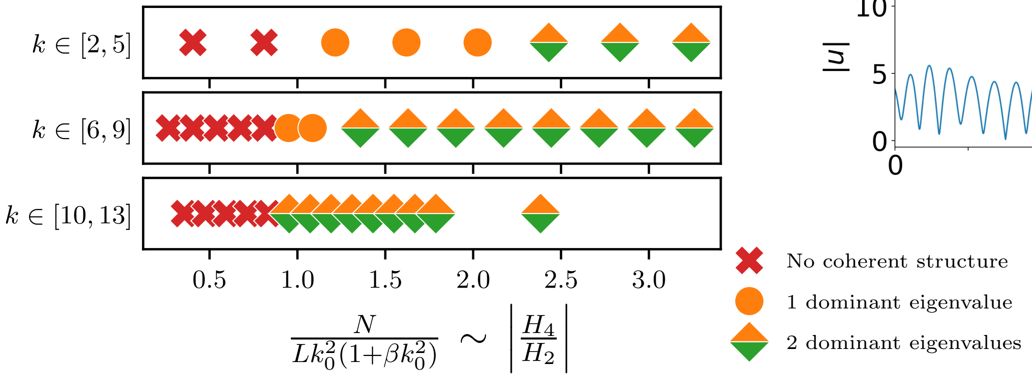
Threshold for bound state formation

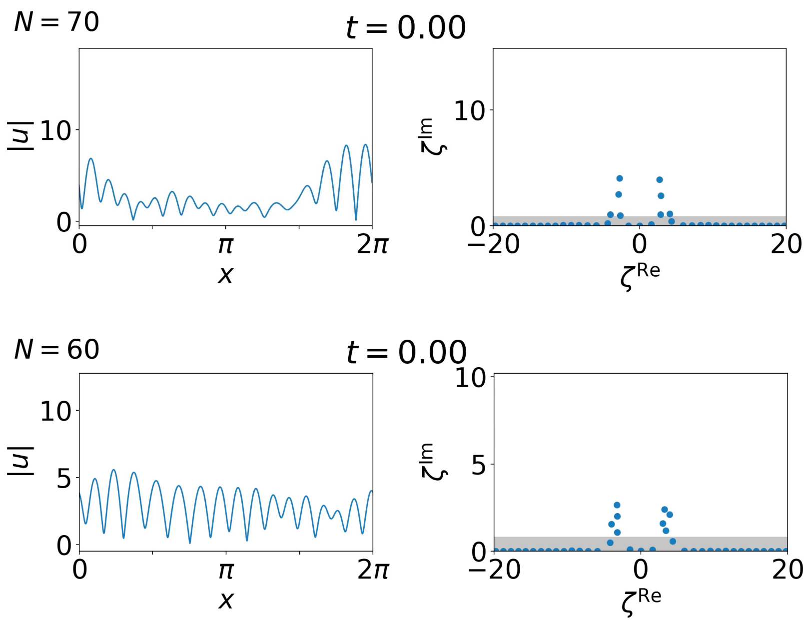
Reducing initial A reduces the amplitude of the eventual bound state, and ζ_1 and ζ_2.

Eventually ζ_2 is no longer distinguishable from the other eigenvalues in the DST spectrum. The dynamics slow down.

Reducing the amplitude further still, no long-lived coherent structure is formed.

Parameter $\sim |H_4/H_2|$ predicts the boundary for formation of coherent structure.





More results

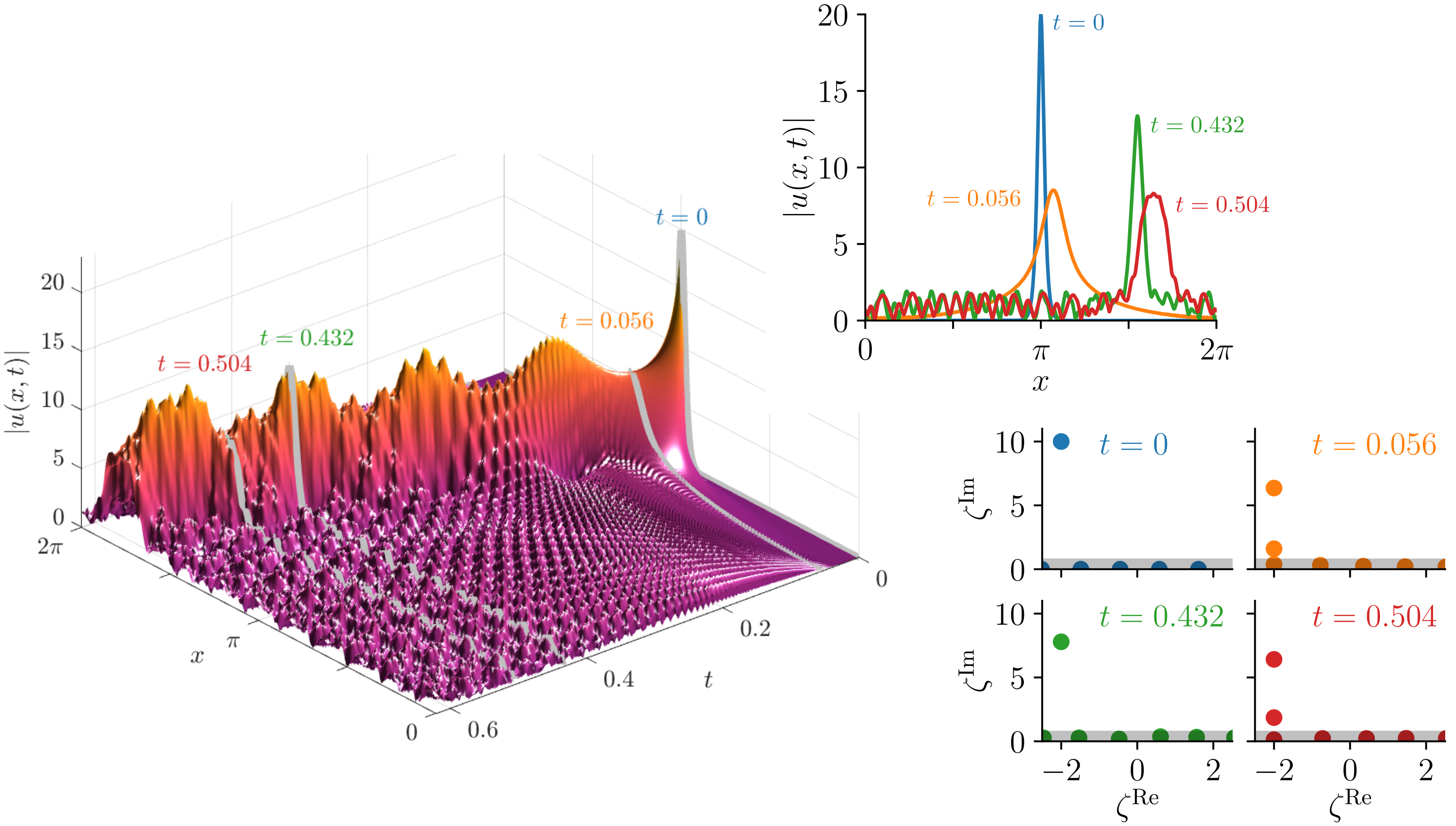
IC 2: single NLSE soliton launched into SHE

To study the formation of the secondary soliton, we initialise with NLSE soliton, with $A = 20$, $v = 4$, $s = L/2$.

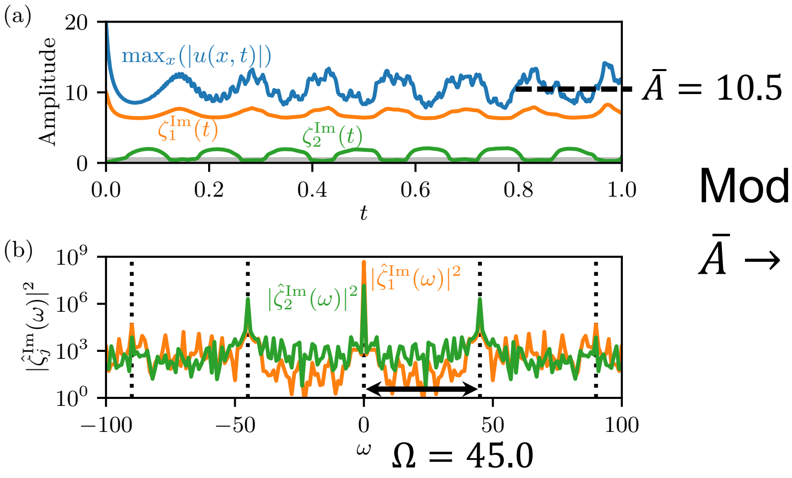
$$u = A\,\mathrm{sech}[A(x - s - vt)]e^{iv(x-s)}e^{i(A^2-v^2)t/2}e^{i\phi}.$$

- Initial profile relaxes by shedding waves, and then recovers and oscillates.
- Lower-amplitude two-solition bound state forms.
- Oscillations in amplitude and ζ_1, ζ_2 are at the same frequency as the rotation frequency of the primary soliton.
- (k, ω) spectrum suggests a resonant wave-primary soliton interaction is responsible for the creation of the secondary soliton.

Spatiotemporal and DST dynamics



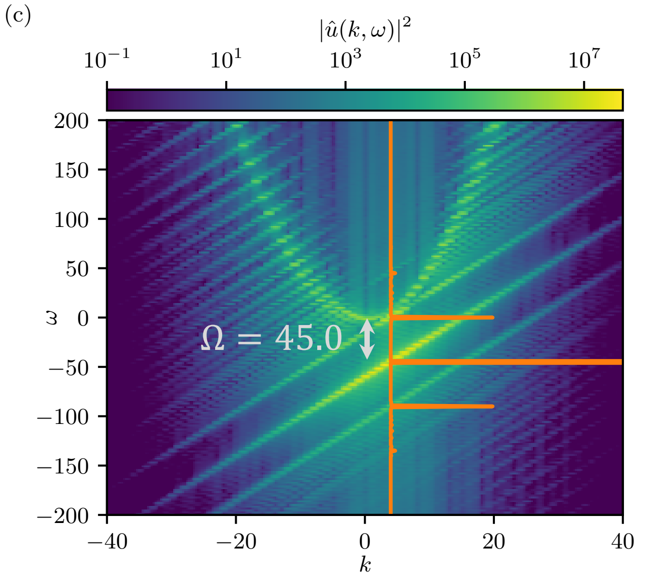
The nature of the bound state



Soliton frequency $\Omega \approx \left(\frac{\bar{A}^2 - v^2}{2}\right) = 47.1$

Modulate amplitude

$\bar{A} \to \bar{A} + 2\Delta\cos(\Omega t), \quad u \sim \bar{A}e^{-i\Omega t} + \Delta + \Delta e^{-2i\Omega t}$



The bound state is well described by an ***NLSE soliton*** *with a* ***modulated amplitude.***

Tempting to suggest that as the bound state absorbs more waves, the secondary trace ***peels off the dispersion relation.***

Yet more results

IC 3: two SHE solitons

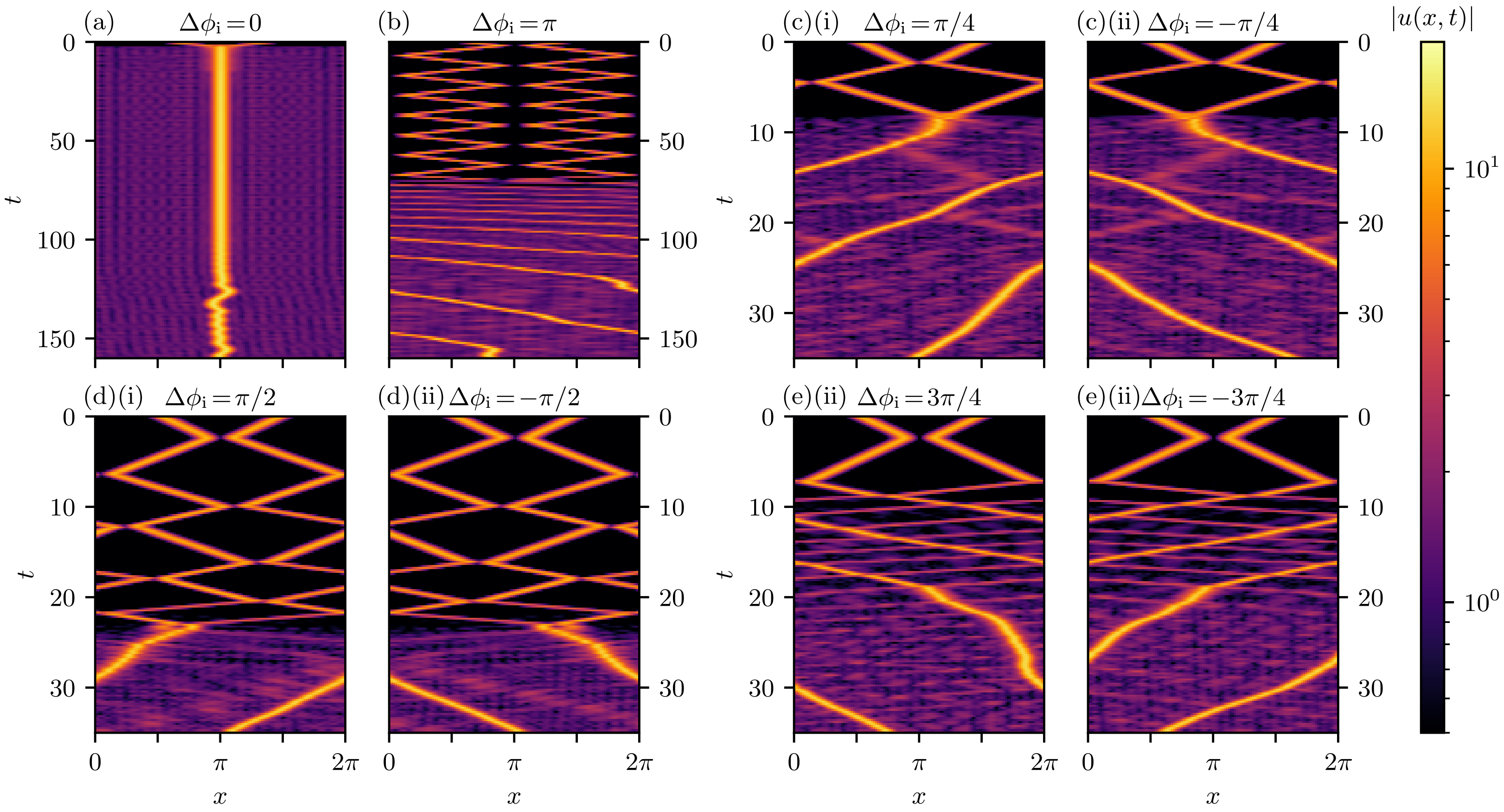
The SHE has its own quasi-soliton solution that balances dispersion with self-focusing (Jia & Lin 2012).

$$u = \frac{3}{\sqrt{8\beta}} \operatorname{sech}^2\left[\frac{1}{\sqrt{4\beta}}(x - vt)\right] e^{ivx} e^{i(1/\beta - v^2)t/2} e^{i\phi}$$

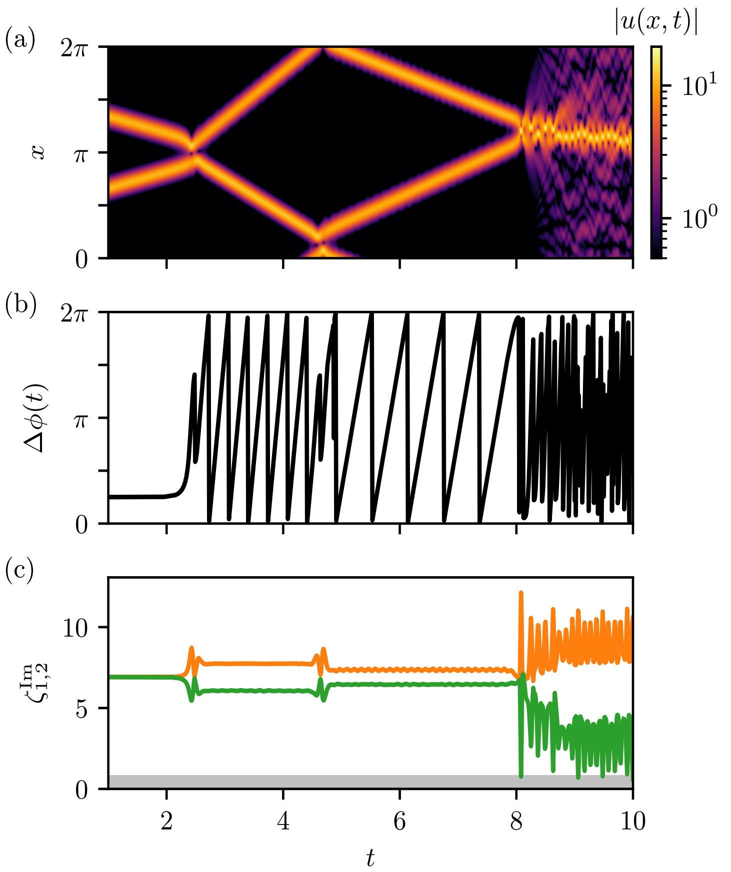
To study soliton mergers, we initialise with two SHE quasi-solitons, with a range of initial phase differences $\Delta\phi = \phi_1 - \phi_2$. ($v = \pm 0.5$)

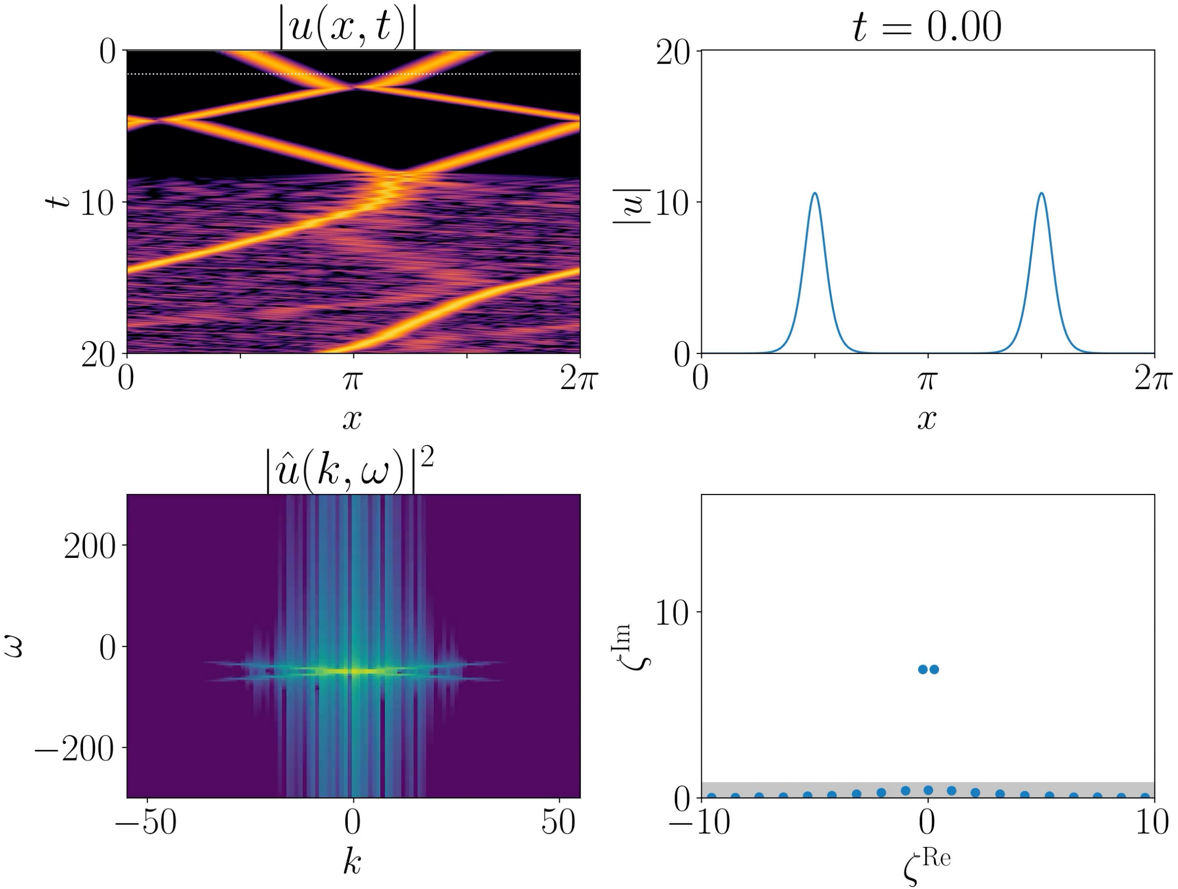
- The solitons undergo numerous inelastic collisions as they re-circulate through the periodic box.
- When they collide, one of them grows at the expense of the other (reflected in the imaginary part of the respective primary eigenvalue).
- Mergers occur when $\Delta\phi = 0$ on collision. One soliton captures another to form the bound state.

Scan in phase difference



Example: $\Delta\phi = \pi/4$





Conclusion and perspectives

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Conclusion and perspectives

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- We have demonstrated the utility of the ***DST as a diagnostic tool*** to examine the weakly nonintegrable Schrödinger-Helmholtz equation.
- We have used it to characterise the ***"statistical attractor"*** that emerges out of strongly nonlinear soliton turbulence, as a ***two-soliton bound state.***
- This ***bound state appears universal,*** as it emerges out of many situations: soliton turbulence, soliton collisions, and soliton-wave interactions.
- We need to understand these mechanisms better mathematically, and to integrate them into the general theory of wave turbulence.
- Other weakly nonintegrable systems are crying out for a similar analysis.

Happy birthday Sergey!

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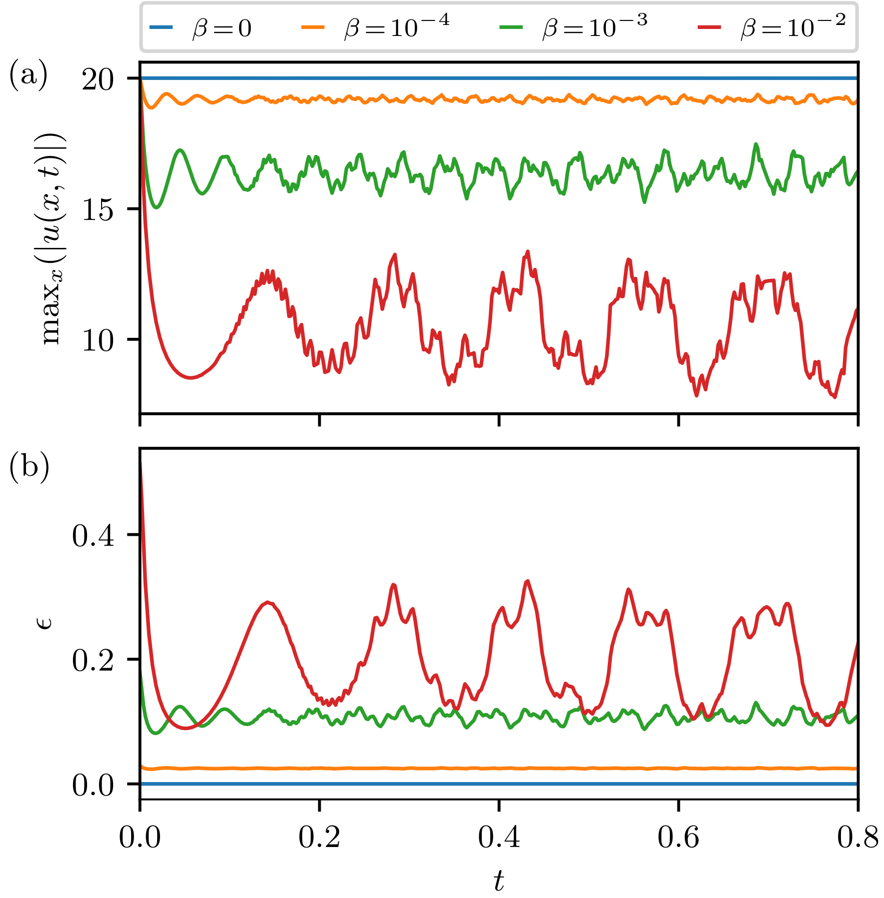
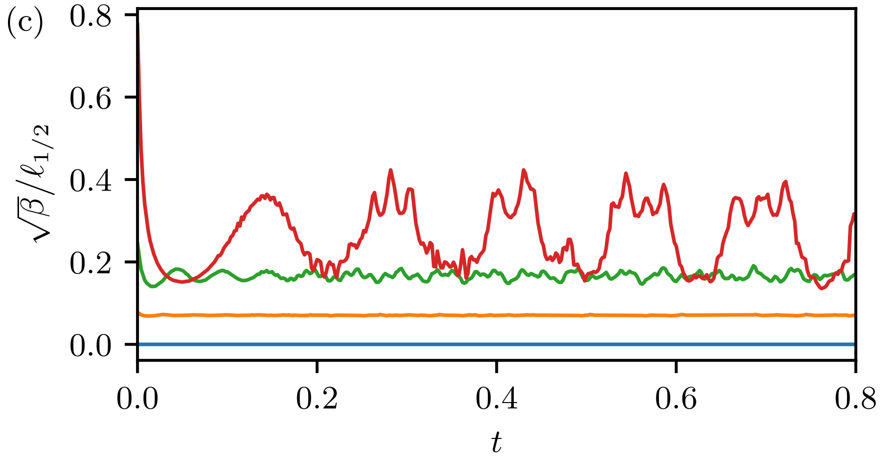
IC 2: NLSE soliton in SHE – scan in β





$\ell_{1/2}$ = FWHM of soliton

$$\epsilon = 1 - H_4/H_4^{NLS}$$