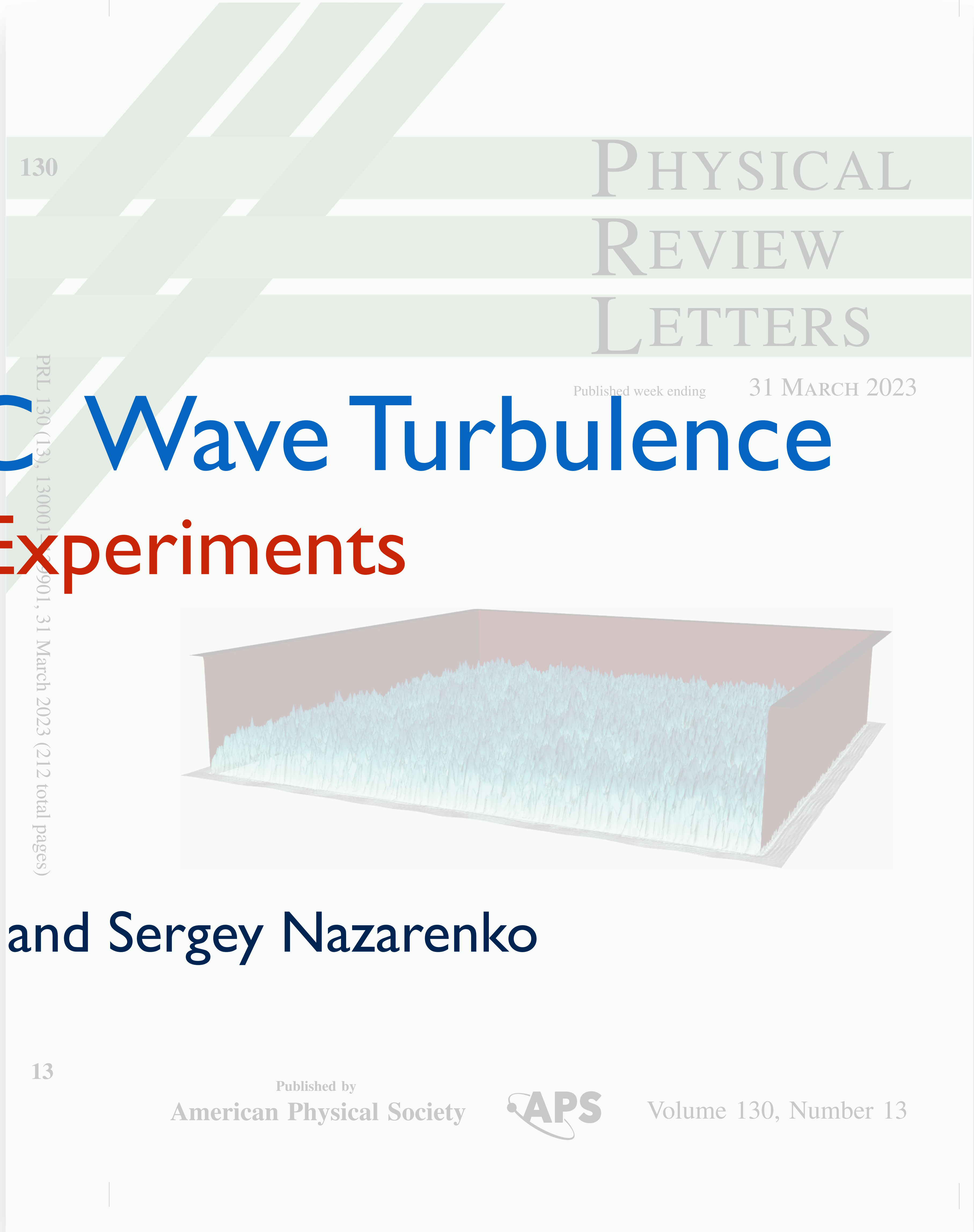
Universal Scaling States in BEC Wave Turbulence

Theories, **Numerics** & Experiments



Ying Zhu collaborating with Giorgio Krstulovic and Sergey Nazarenko

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Université de Côte d'Azur, CNRS,
Nice, France

Gross-Pitaevskii Equation (AKA: Nonlinear Schrödinger Equation)

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + U(\mathbf{r})\,\psi(\mathbf{r},t) + g\,|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t)$$

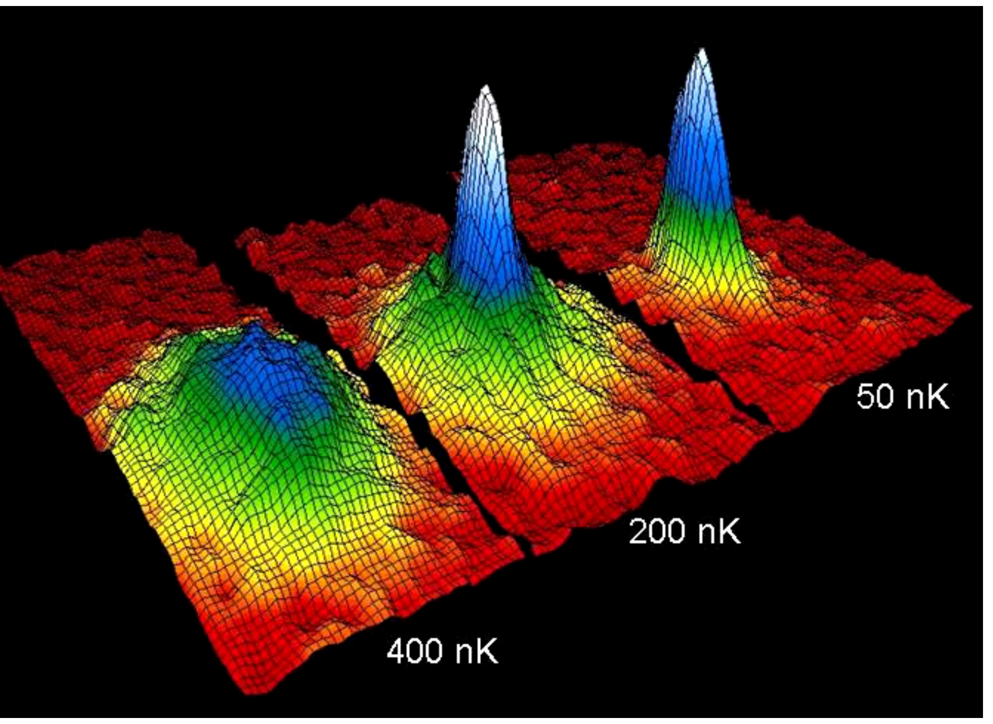
$$g = \frac{4\pi a\hbar^2}{m}$$

One simple equation for many physical systems

$g > 0$, defocusing case

$g < 0$, focusing case

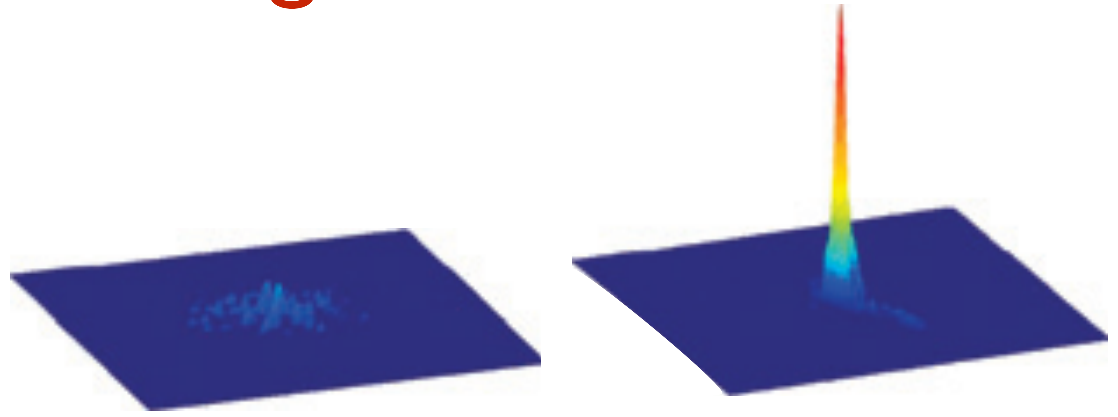
Bose-Einstein condensates (BEC)



E. A. Cornell and C. E. Wieman, 95

Classical light "condensates"

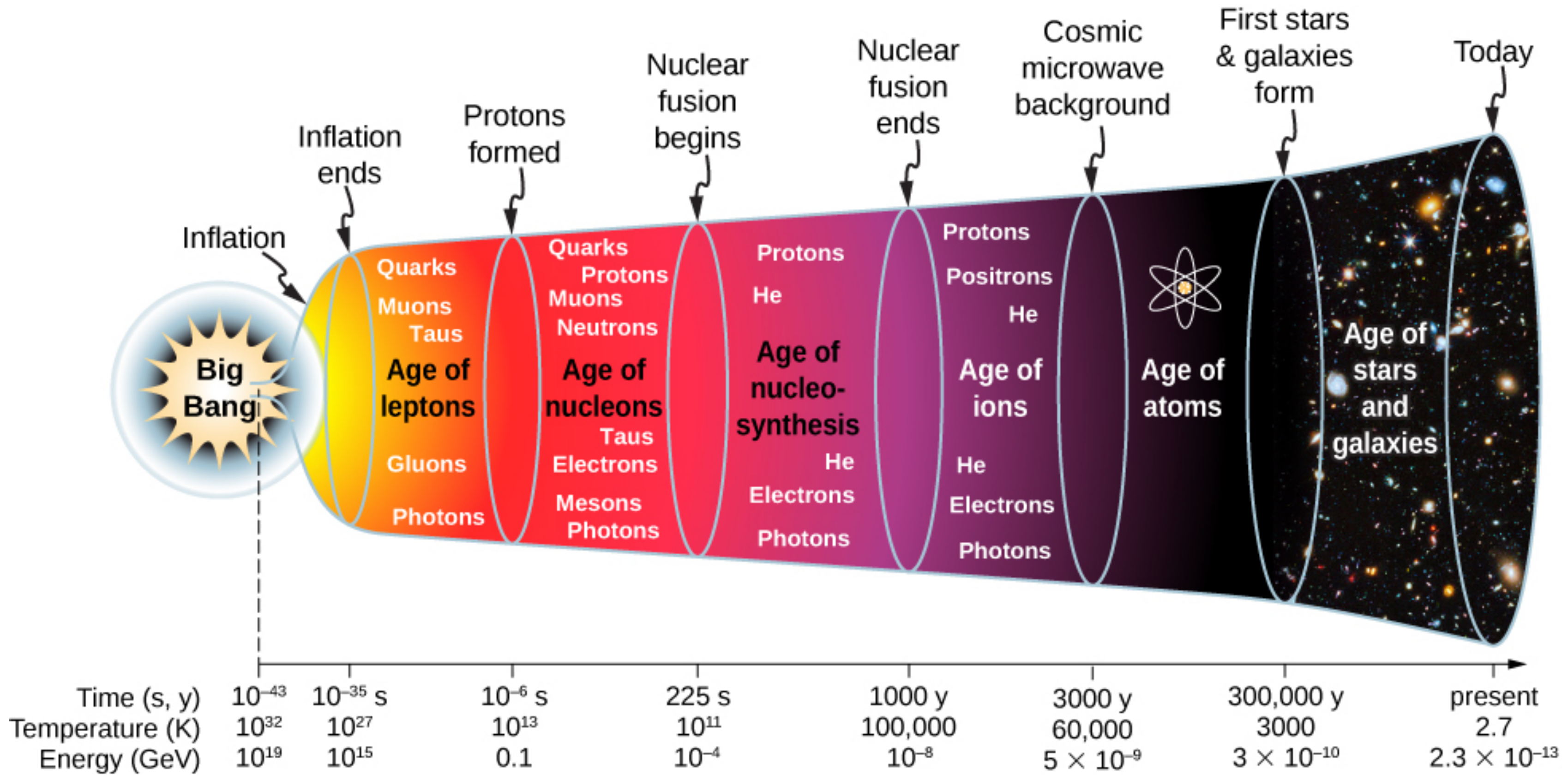
Fourier space (k)



E/N (m^{-1}) = 13.3 1.8

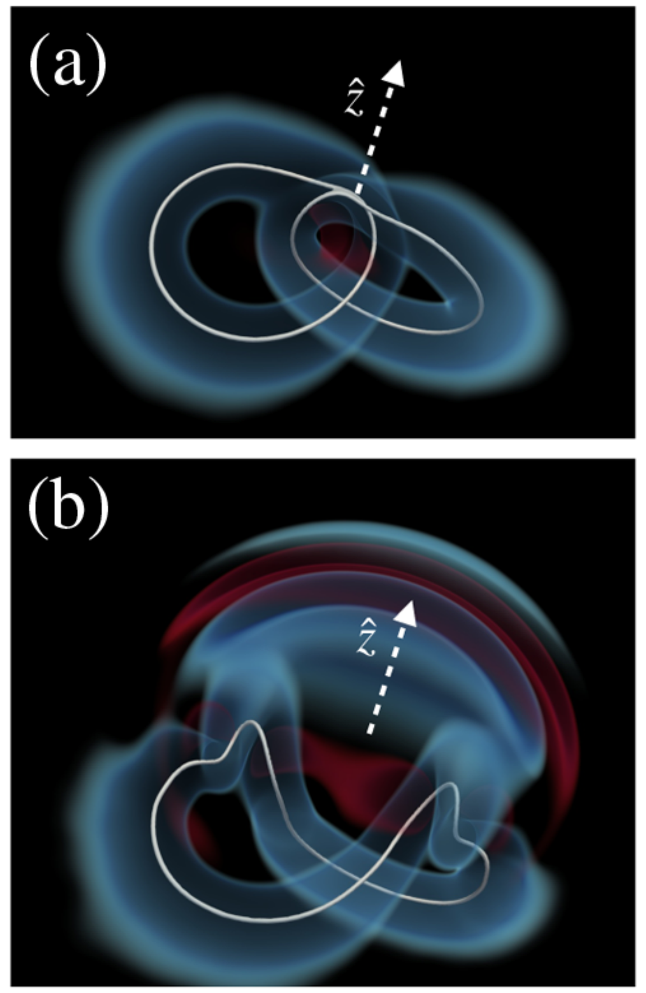
Sun et al, Nature Physics 2012.
Klaers et al, Nature 2010.

A symmetry breaking epoch between 10^{-12} and 10^{-6} seconds

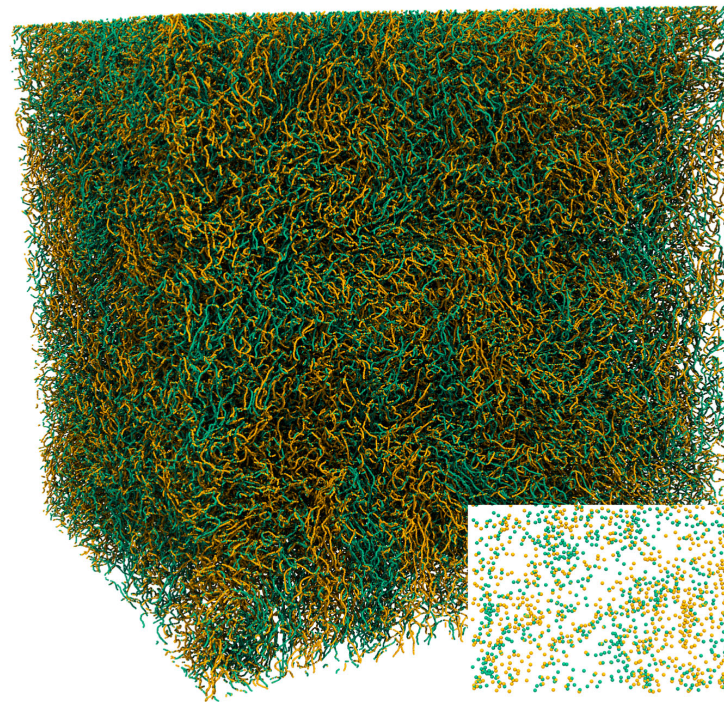


W.H. Zurek, Physics Reports (1996)

Vortex dynamics and Kolmogorov (strong) turbulence



Villois, Proment, Krstulovic. Phys. Rev. Letts. **125**, 164501 (2020)



Polanco, Müller, Krstulovic Comms. (2021)12:709

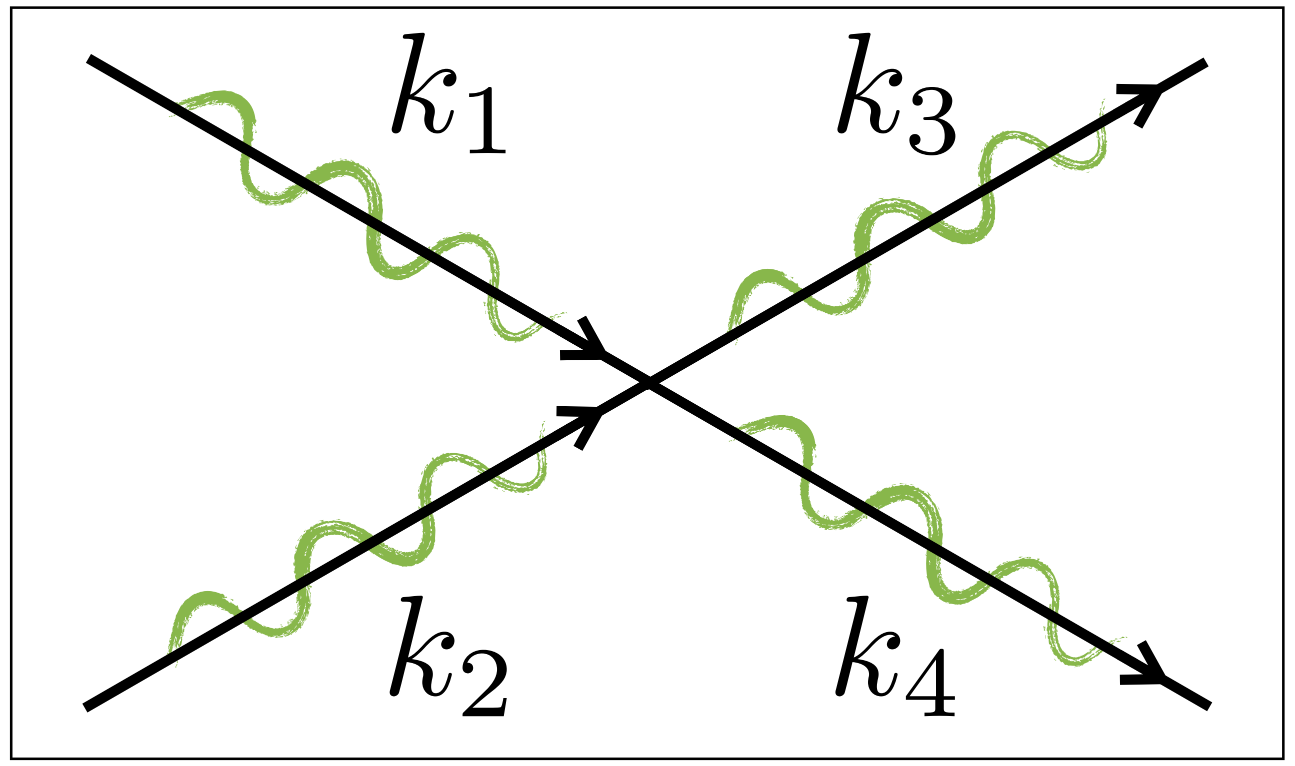
Nonlinear wave interaction in BEC

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi,$$

"free particles"

Matter waves : $\omega_k = \frac{\hbar}{2m}k^2$

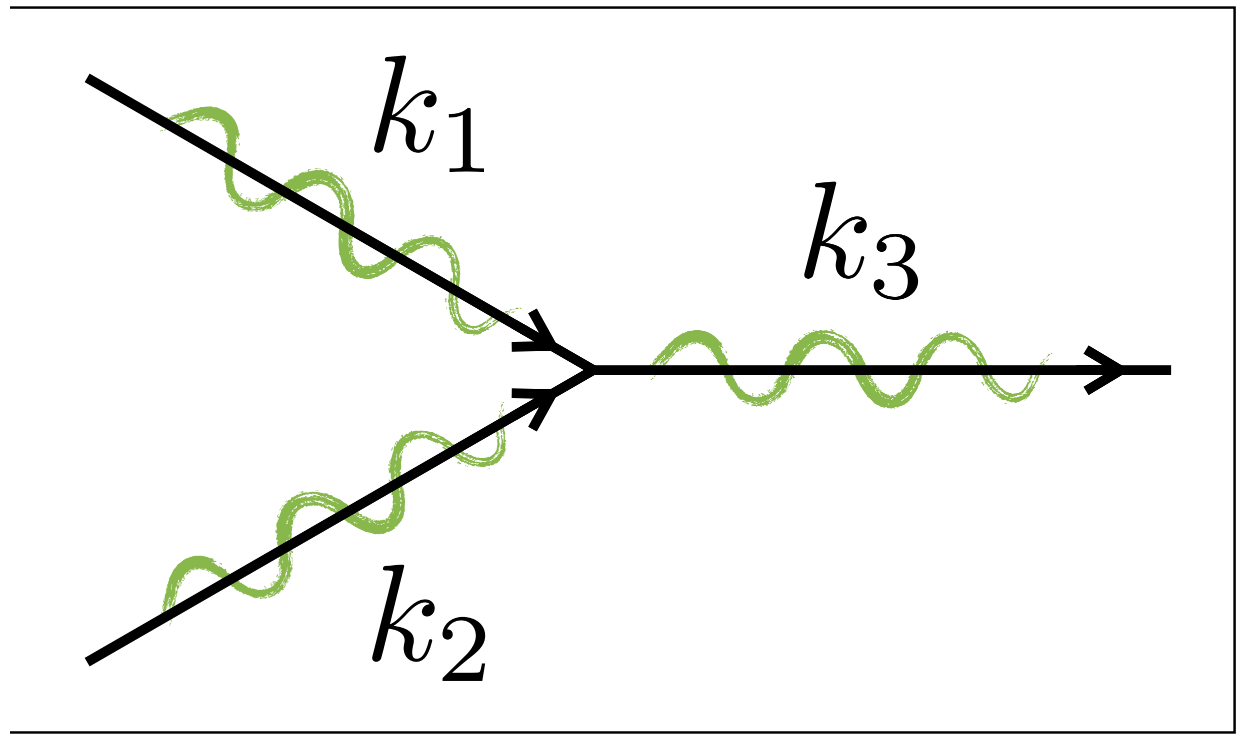
$\psi = 0 + \delta\psi$ → Cubic nonlinearity



Strong background condensate

Bogoliubov waves (phonons) : $\omega_k = c_s k\sqrt{1+\xi^2k^2/2}$

$\psi = A_0 + \delta\psi$ → Quadratic nonlinearity



Two types of density waves

acoustic modes : $\omega_k = c_s k$, for $k\xi \ll 1$

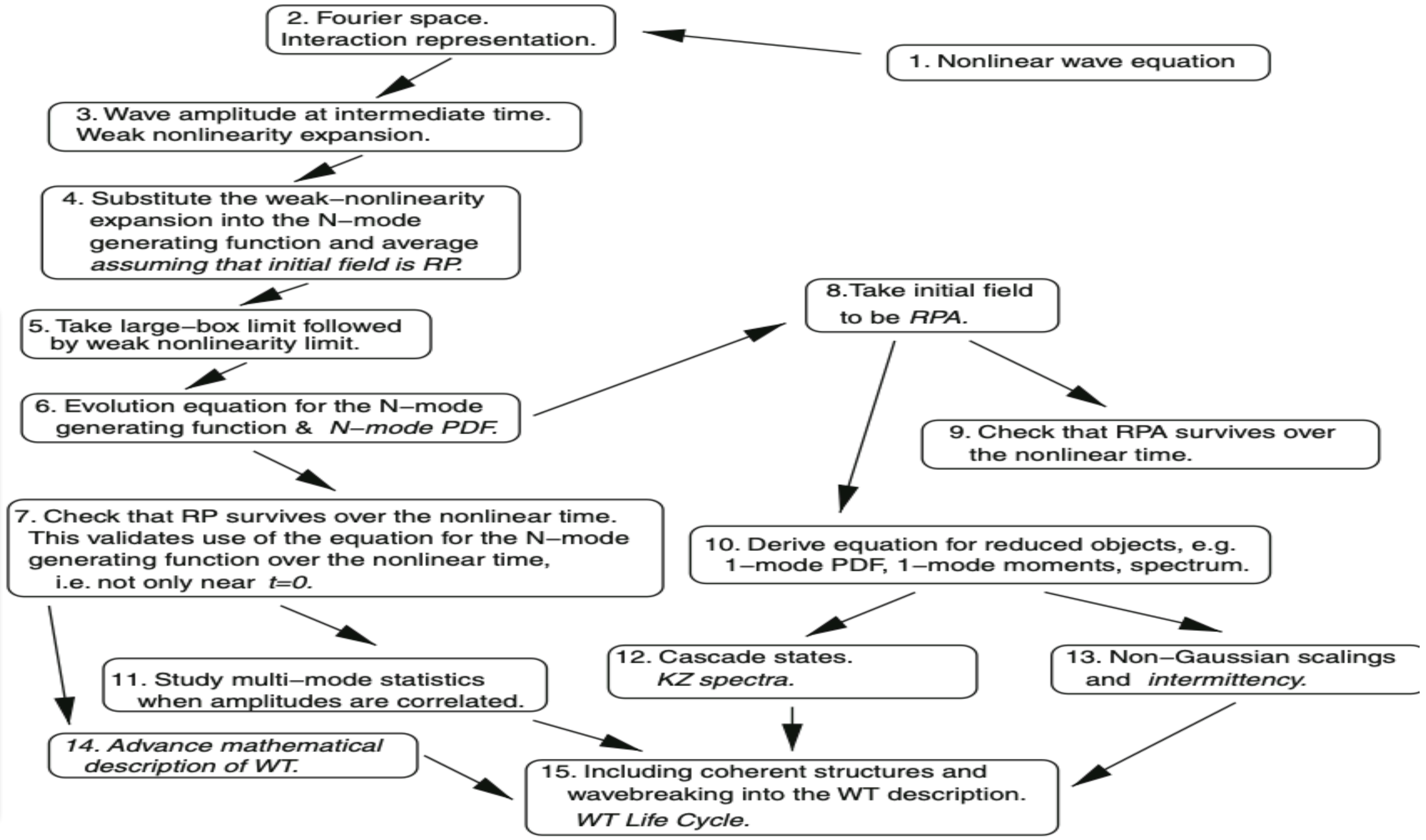
short-wave modes: $\omega_k = c_s\xi/\sqrt{2}\,k^2$, for $k\xi \gg 1$

Wave Turbulence Theory (WWT)

WWT: mathematical framework describing the statistical behaviour of WT dominated by weakly nonlinear waves.

Stage 1: deriving the wave-kinetic equation (WKE), and/or equation for 1-mode PDF

Stage 2: analysis based on above equations, KZ spectra, non-stationary evolution, joint PDF, …



WWT formalism

Wave-kinetic equation: evolution of the wave-action spectrum

similar to Boltzmann Equation

second-order moment of the wave amplitude

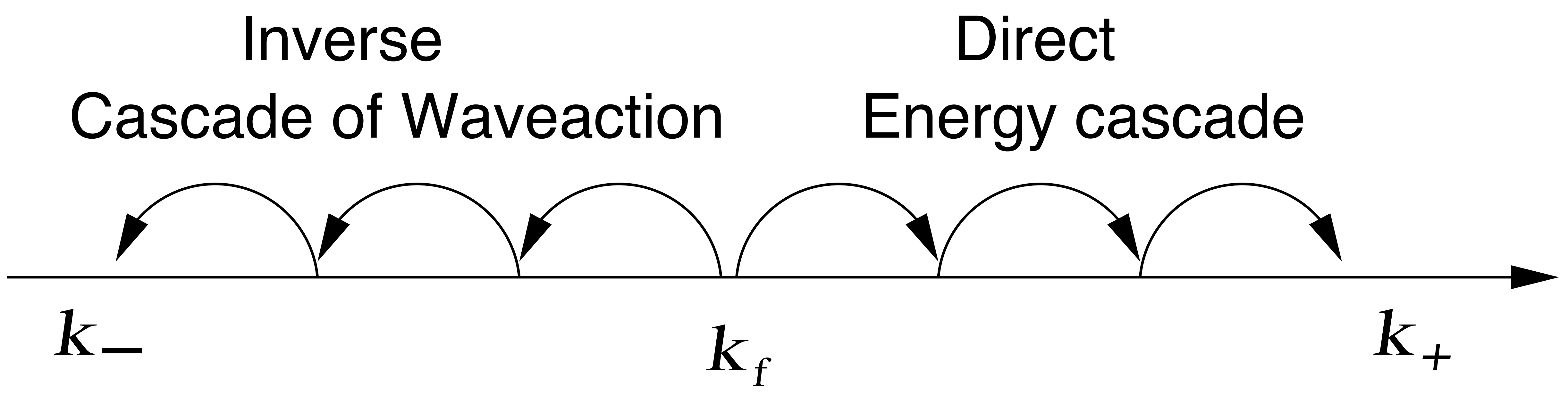
Condition to apply WWT: separation of spatial scales and time scales

Towards strong waves: critical balance, wave breaking …

Four-wave regime: dual cascades

Two invariants $N = \int 4\pi k^2 n_k,$ $E = \int 4\pi k^2 \omega_k n_k = \int 4\pi k^4 n_k$

- Like in 2D Euler, the ratio of densities of the two invariants is k^2
- Mapping 2D Euler to GPE invariants: $E \to N, \quad \Omega \to E\,.$

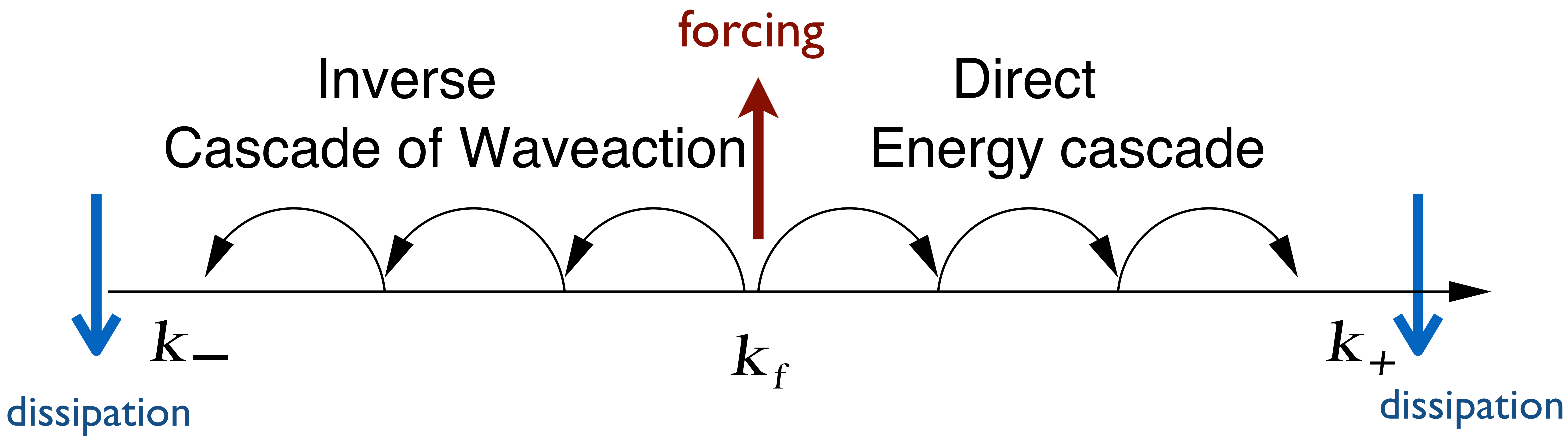


Dual cascade in wave turbulence turbulence

In GPE waveaction is *N*

$$\frac{dn_k}{dt} = \frac{32\pi^3}{k} \int \min(k, k_1, k_2, k_3)\delta^{23}_{1\omega} n_k n_1 n_2 n_3 \left(\frac{1}{n_k} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right) k_1 k_2 k_3 dk_1 dk_2 dk_3.$$

Forcing and dissipation setup

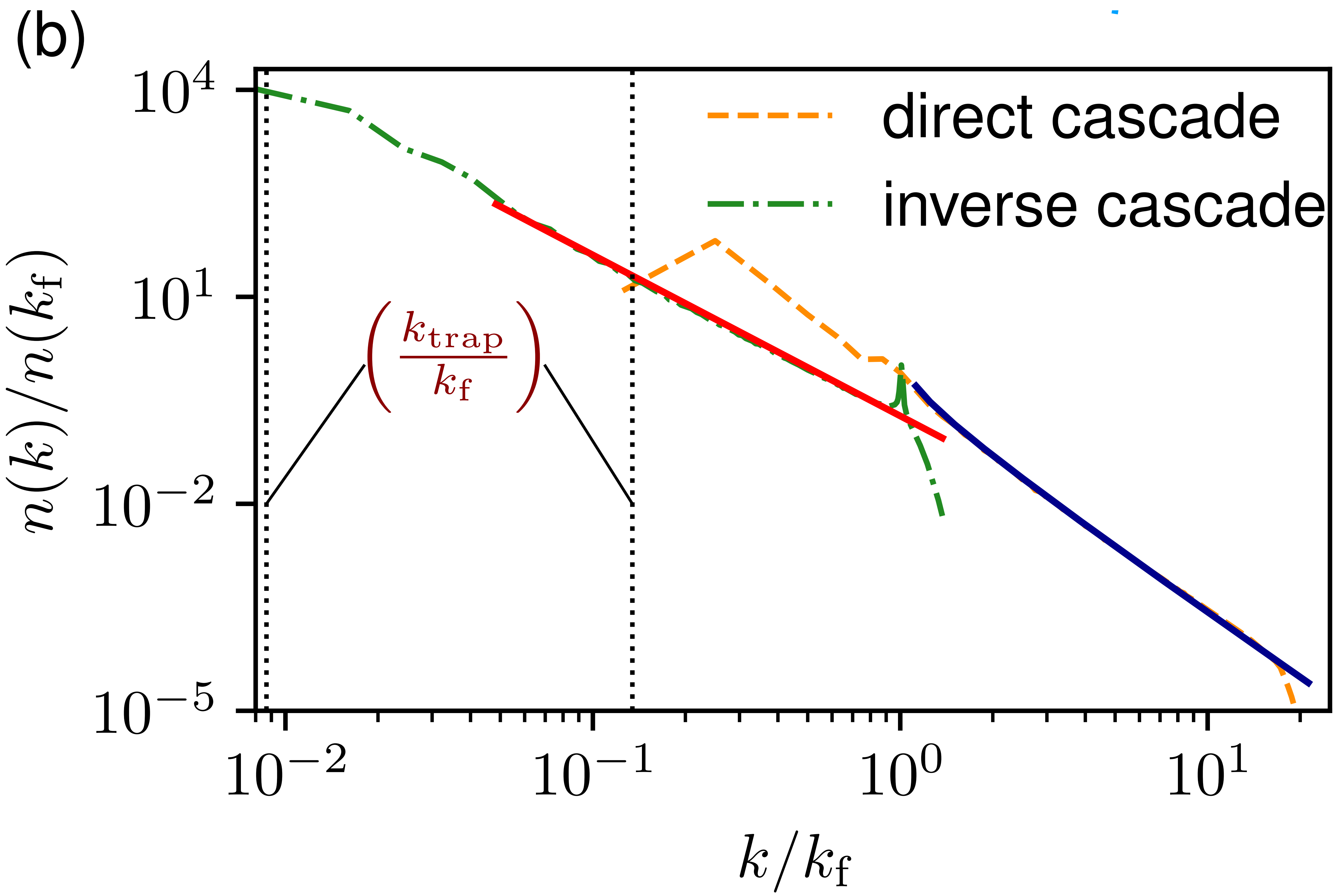


One cascade can not live without the other one !

BEC wave turbulence in a trap

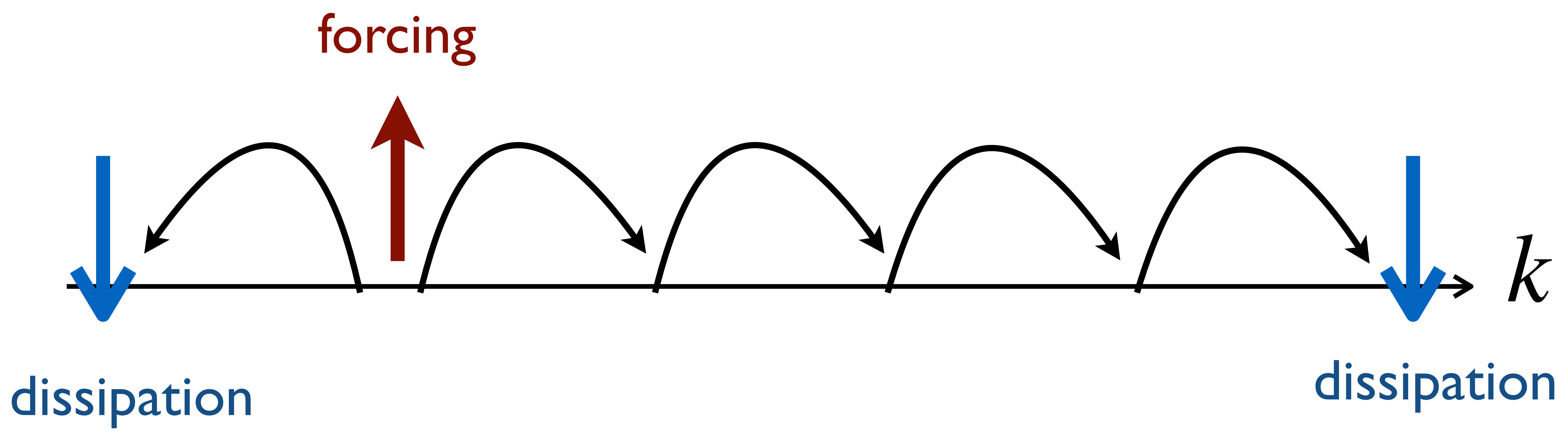
$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U(\mathbf{x})\psi + g|\psi|^2\psi$$

+ forcing + dissipation

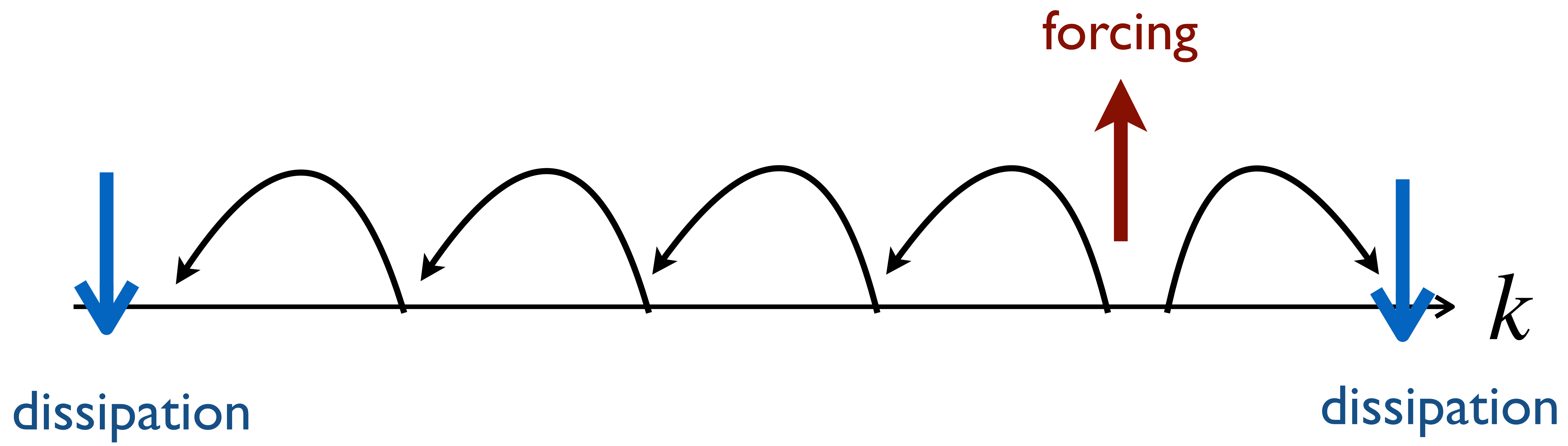


Forcing and dissipation setup

Steady energy cascade



Steady particle cascade

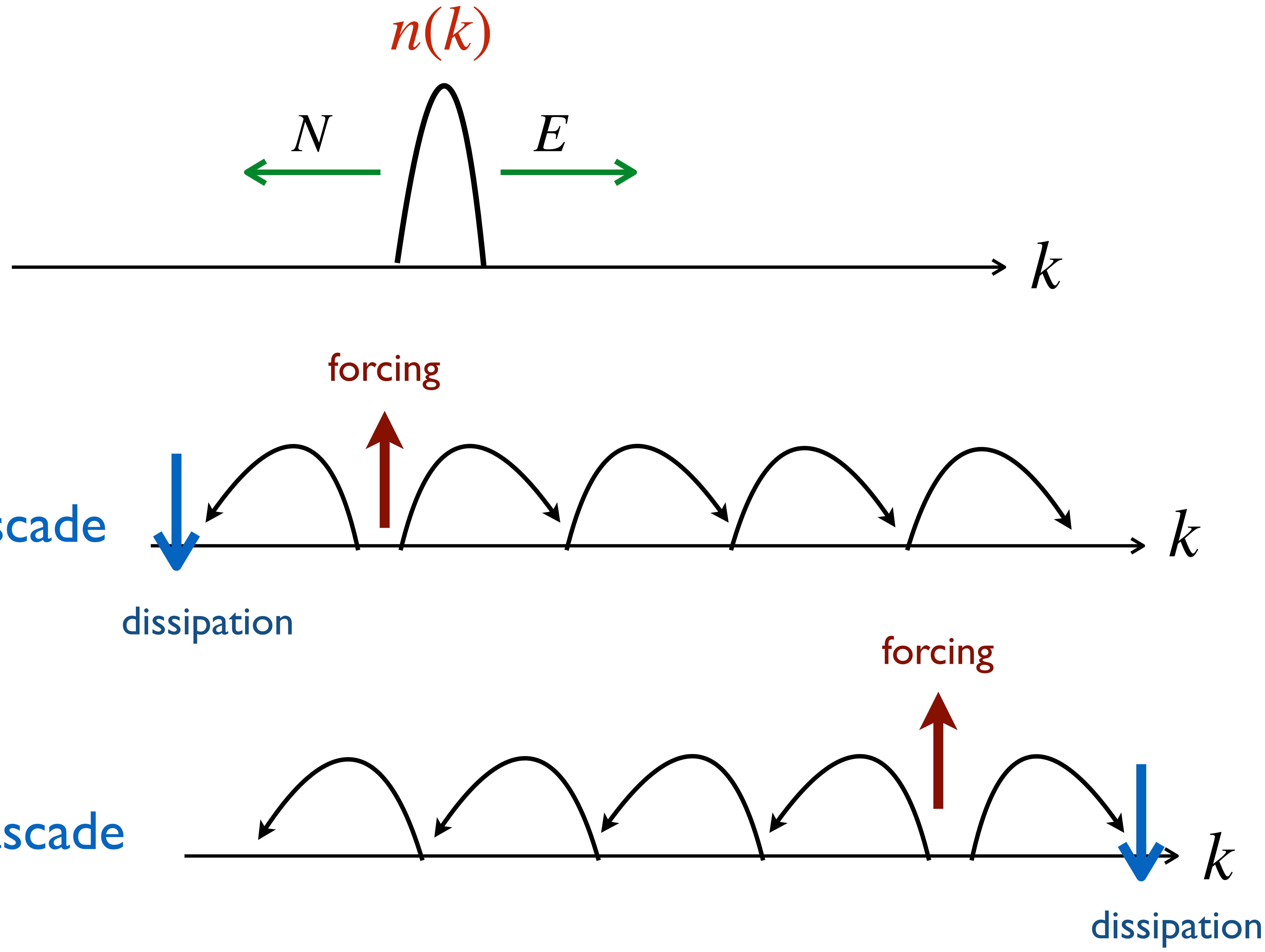


Forcing and dissipation setup

Free-decay

Forced energy cascade

Forced particle cascade



Advanced GPE simulations for wave turbulence dynamics

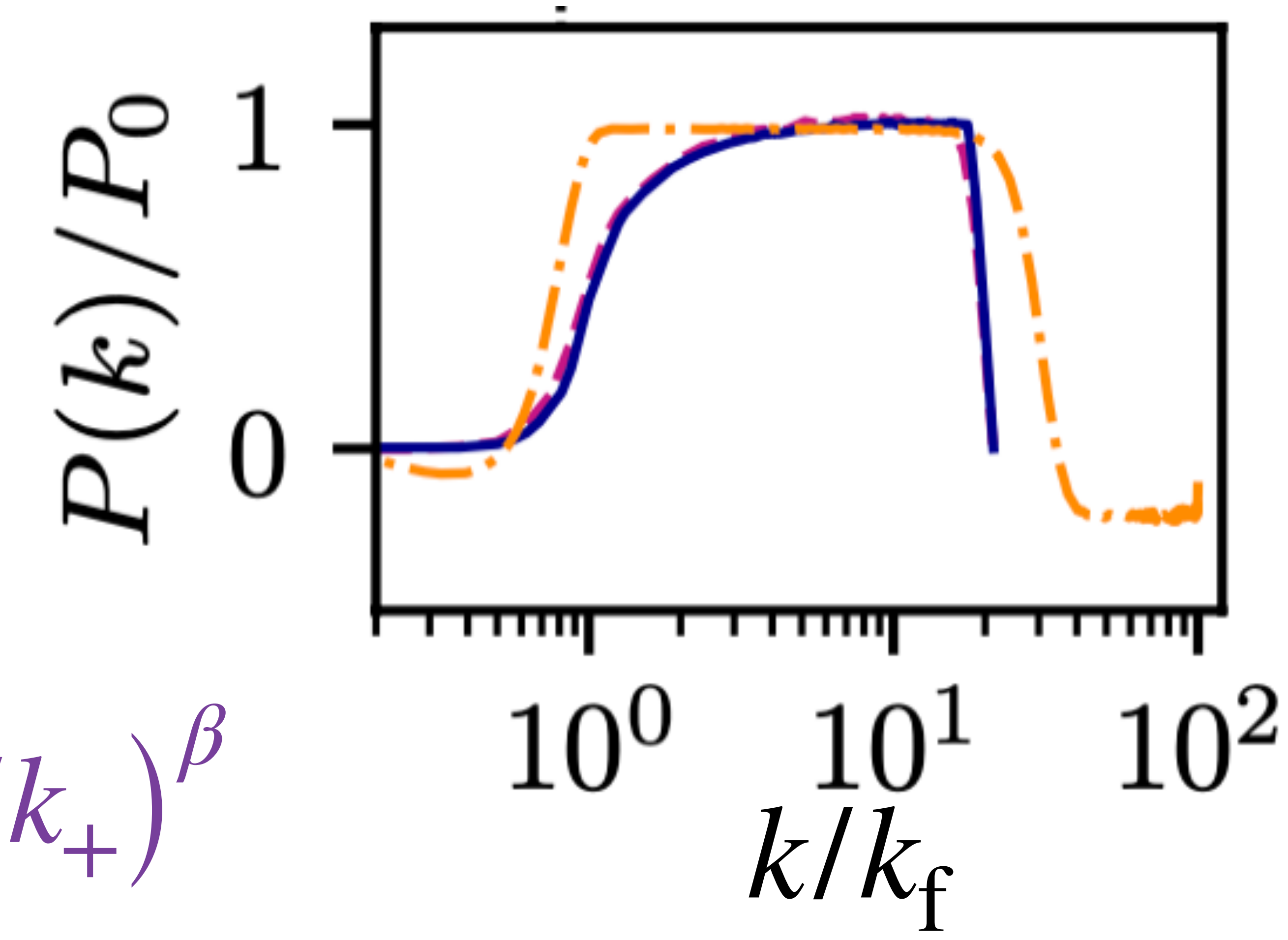
$$i\frac{\partial\hat{\psi}_{\boldsymbol{k}}(t)}{\partial t} = k^2\hat{\psi}_{\boldsymbol{k}}(t) + \sum_{\boldsymbol{k}+\boldsymbol{k}_1=\boldsymbol{k}_2+\boldsymbol{k}_3}\hat{\psi}^*_{\boldsymbol{k}_1}(t)\hat{\psi}_{\boldsymbol{k}_2}(t)\hat{\psi}_{\boldsymbol{k}_3}(t) + iF_{\boldsymbol{k}} - iD_{\boldsymbol{k}}\hat{\psi}_{\boldsymbol{k}}(t)$$

- Pseudo-spectral method .vs. finite difference scheme

$$\sum_{\boldsymbol{k}+\boldsymbol{k}_1=\boldsymbol{k}_2+\boldsymbol{k}_3}\hat{\psi}^*_{\boldsymbol{k}_1}(t)\hat{\psi}_{\boldsymbol{k}_2}(t)\hat{\psi}_{\boldsymbol{k}_3}(t) \rightarrow |\psi(\boldsymbol{x},t)|^2\psi(\boldsymbol{x},t)$$ triply-periodic cube

- De-aliasing and conservation, clean flux
- Stochastic forcing

$$\mathrm{d}F_{\boldsymbol{k}}(t) = -\gamma\hat{\psi}_{\boldsymbol{k}}dt + f_0\,\mathrm{d}W_{\boldsymbol{k}}\,, \qquad N \sim t\,, E \sim t^2$$



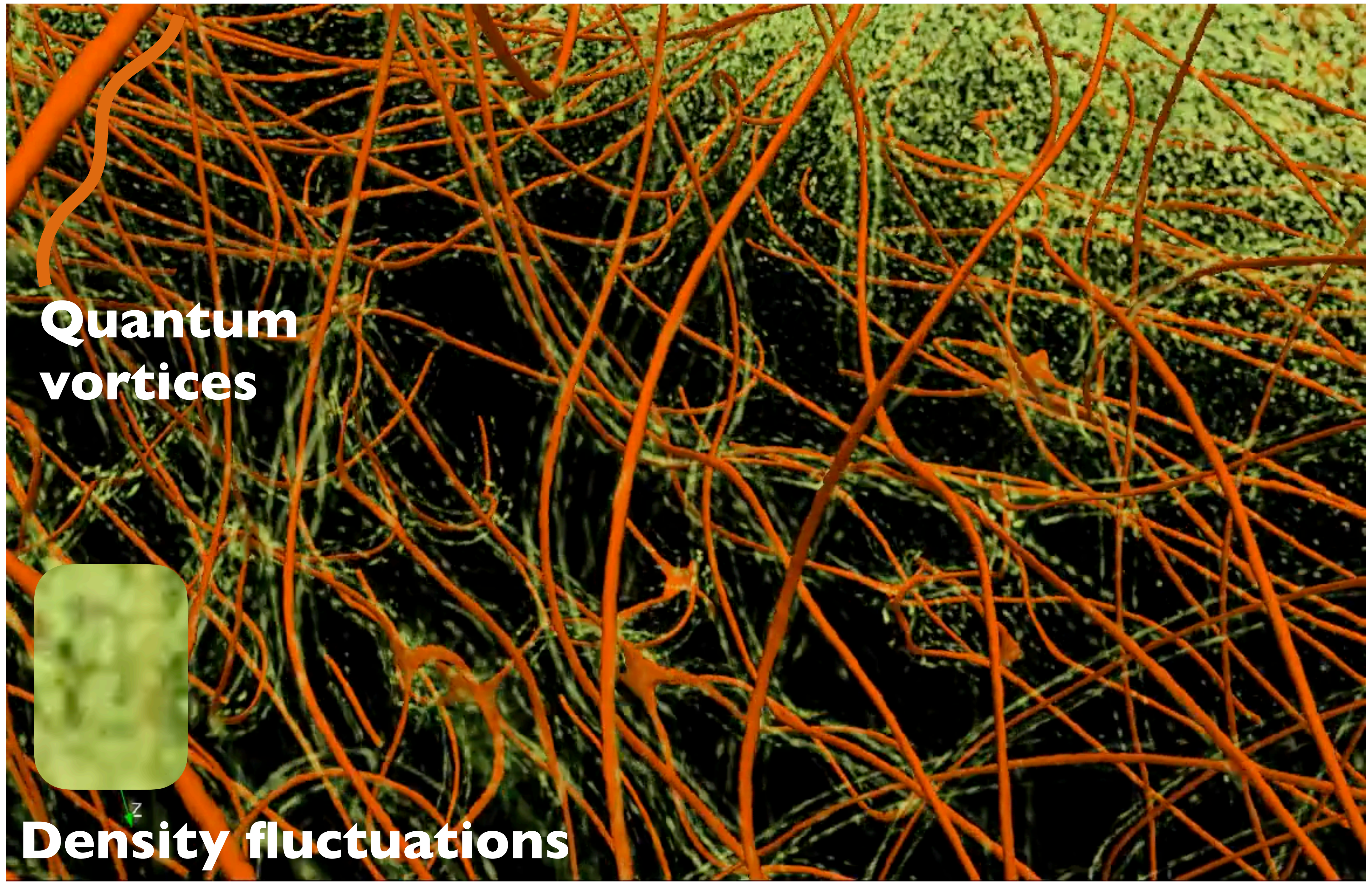
- Hyper-viscosity and hypo-viscosity $D_k = (k/k_-)^{-\alpha} + (k/k_+)^{\beta}$
- Exponential Runge-Kutta temporal scheme, stiff system

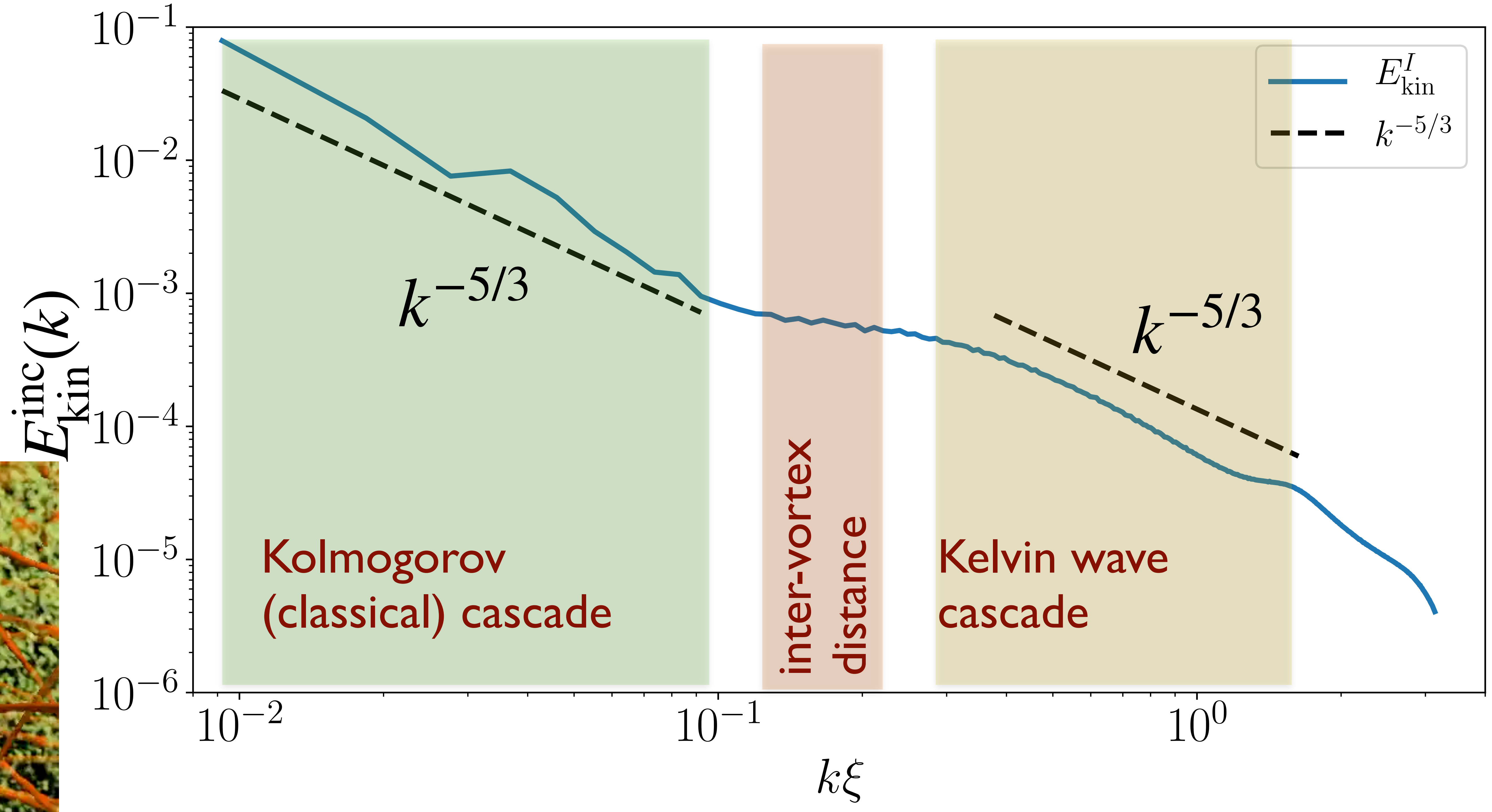
$$\mathrm{d}t \ll 1/\max(k^2, D_k) \longrightarrow \mathrm{d}t \sim 1/\max(k^2, D_k)$$

- High-resolution, massively-parallel code with MPI/OpenMP $512^3 - 1536^3$

FROST code: powerful tool for simulating GPE

- Vortices, vortex tracking 1024^3
- 3D Kolmogorov turbulence 2048^3
- 2D Kolmogorov turbulence 8192^2





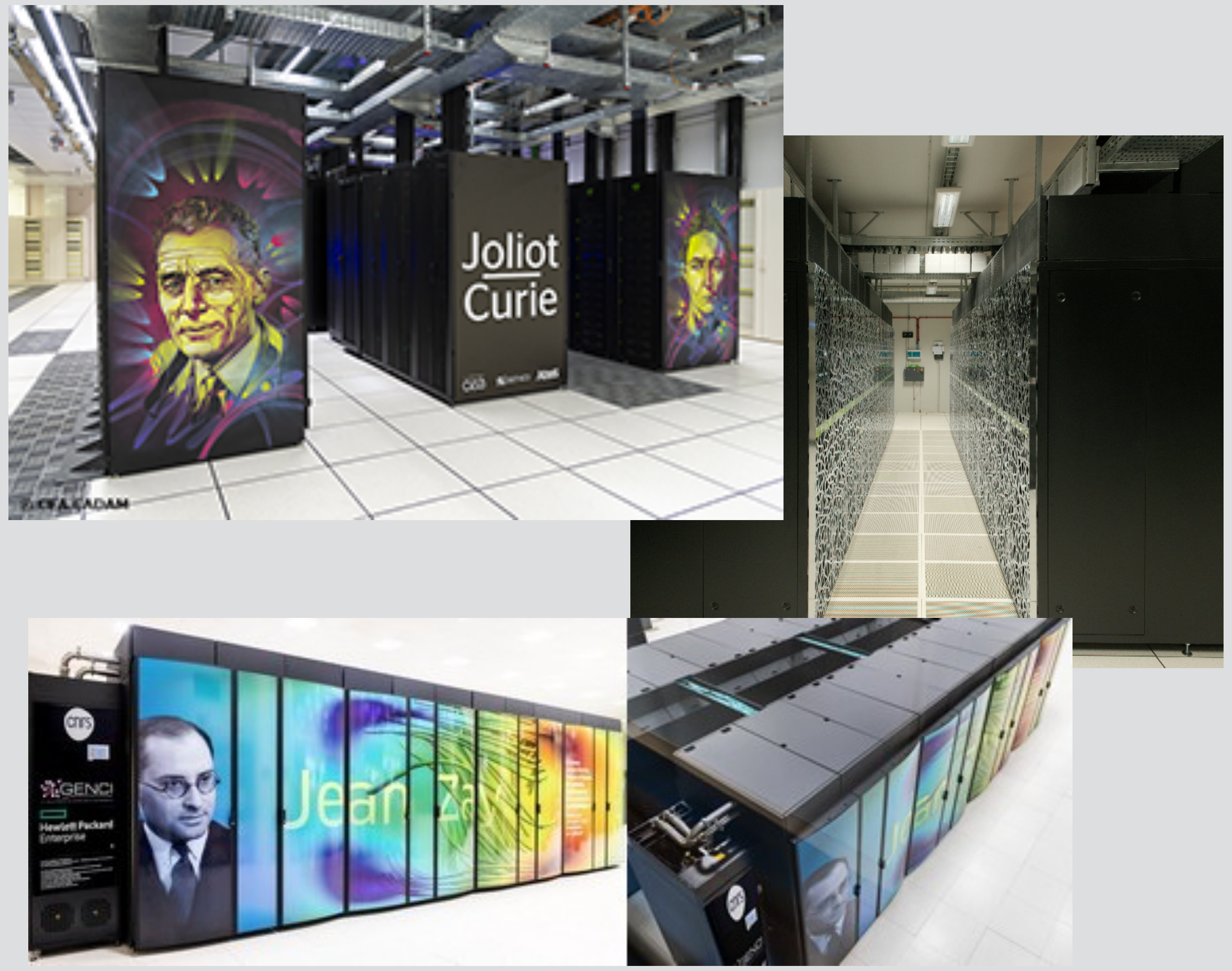
Non-local high-order nonlinearity GPE

Müller, N. P., & Kr stulovic, G. Phys. Rev. B 102, 134513 (2020)

Polanco, J. I.,Müller, N. P., & Kr stulovic, G. Nature Communications, 12(1), 7090 (2021)

Müller, N. P., & Kr stulovic, G. Phys. Rev. Lett. 132, 094002. (2024)

GPE simulation .vs. WKE simulation



GENCI
Le calcul intensif au service de la connaissance

3D GPE + forcing + dissipation
WKE + forcing + dissipation

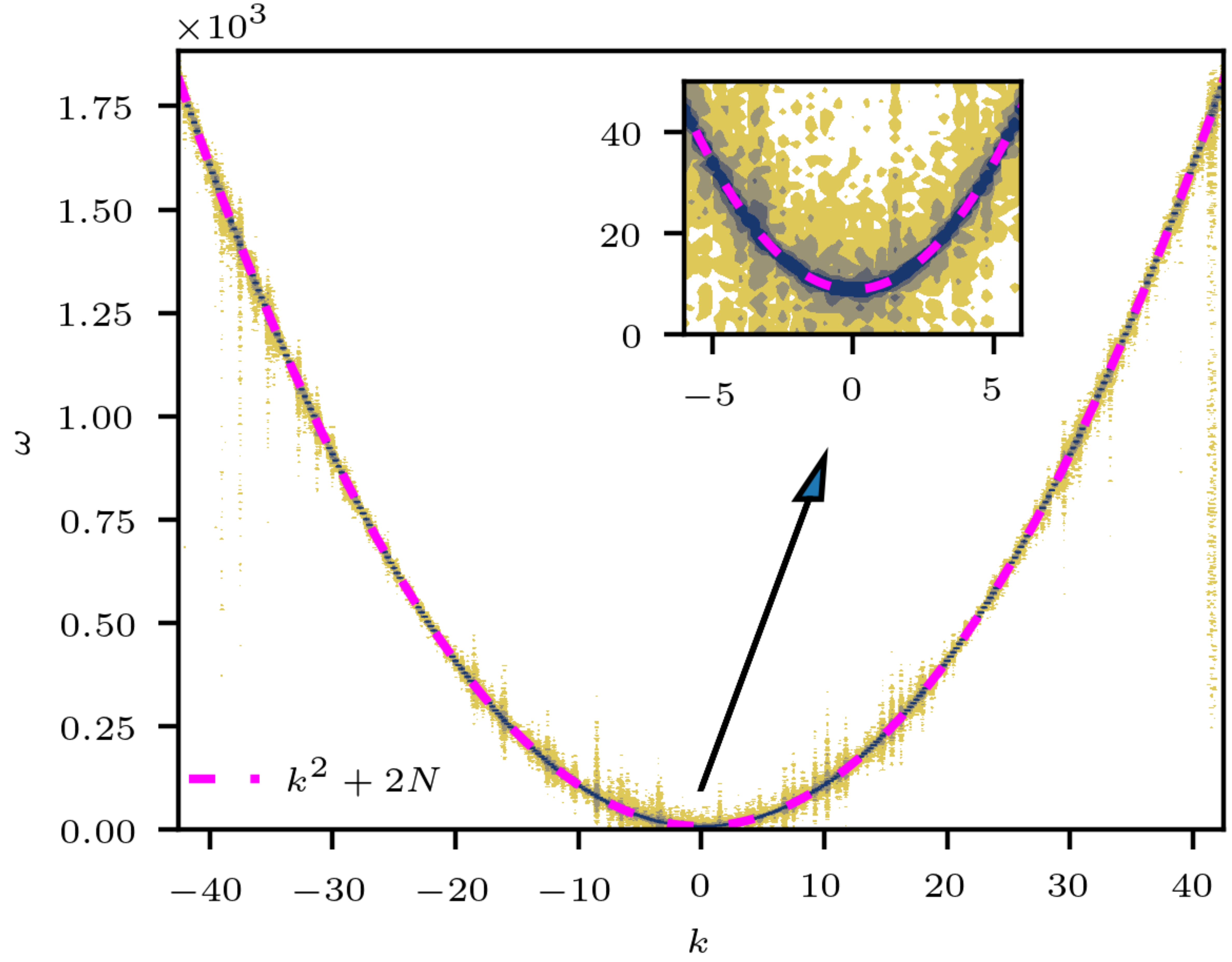
Simulating WKE
- Quick and precise test for theoretical derivation
- Inspire new solutions

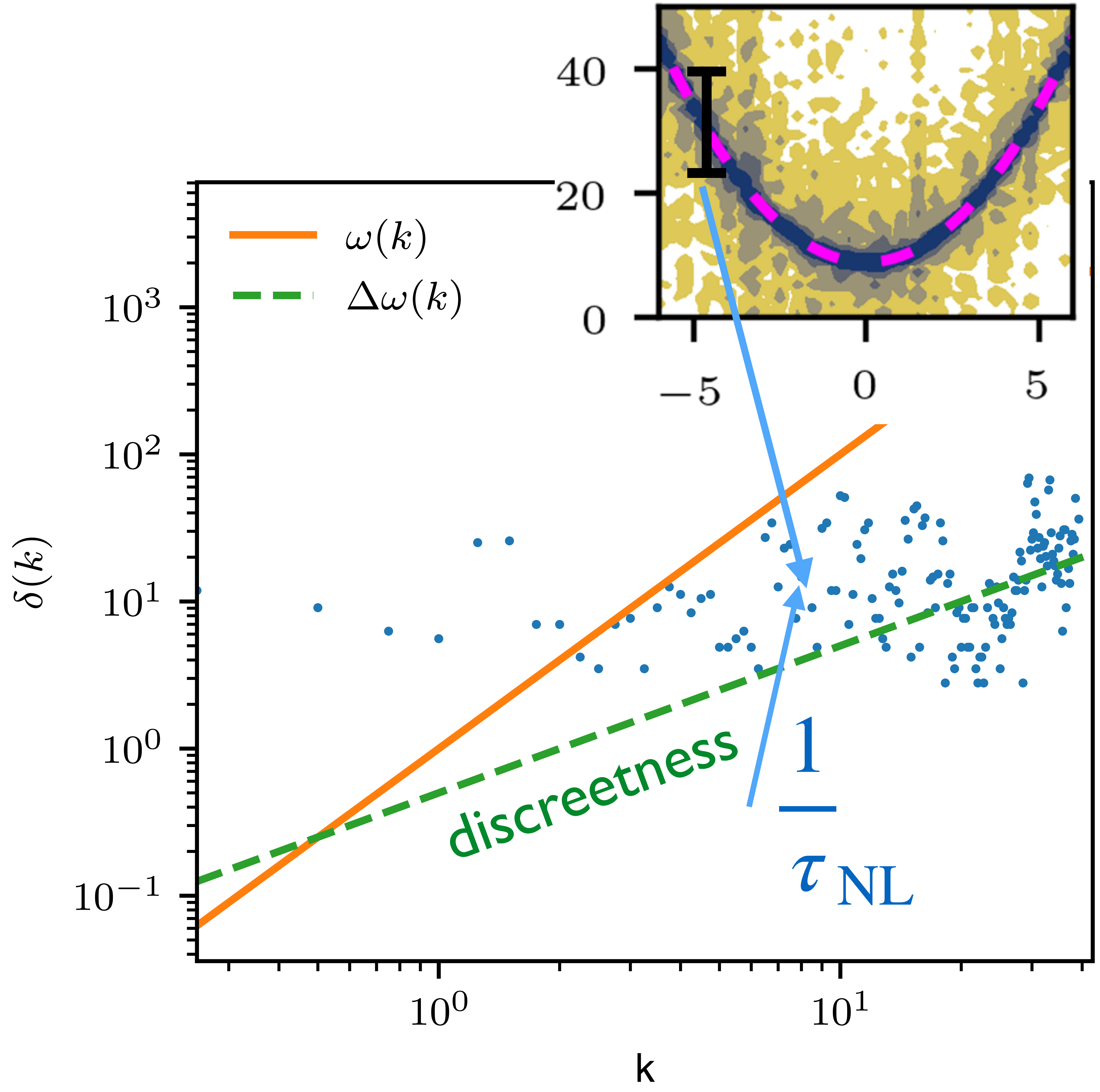
How to "see" waves

$\psi(\boldsymbol{x}, t) =$ "random waves" → $\hat{\psi}(\boldsymbol{k}, \omega)$ → $|\hat{\psi}(\boldsymbol{k}, \omega)|^2$

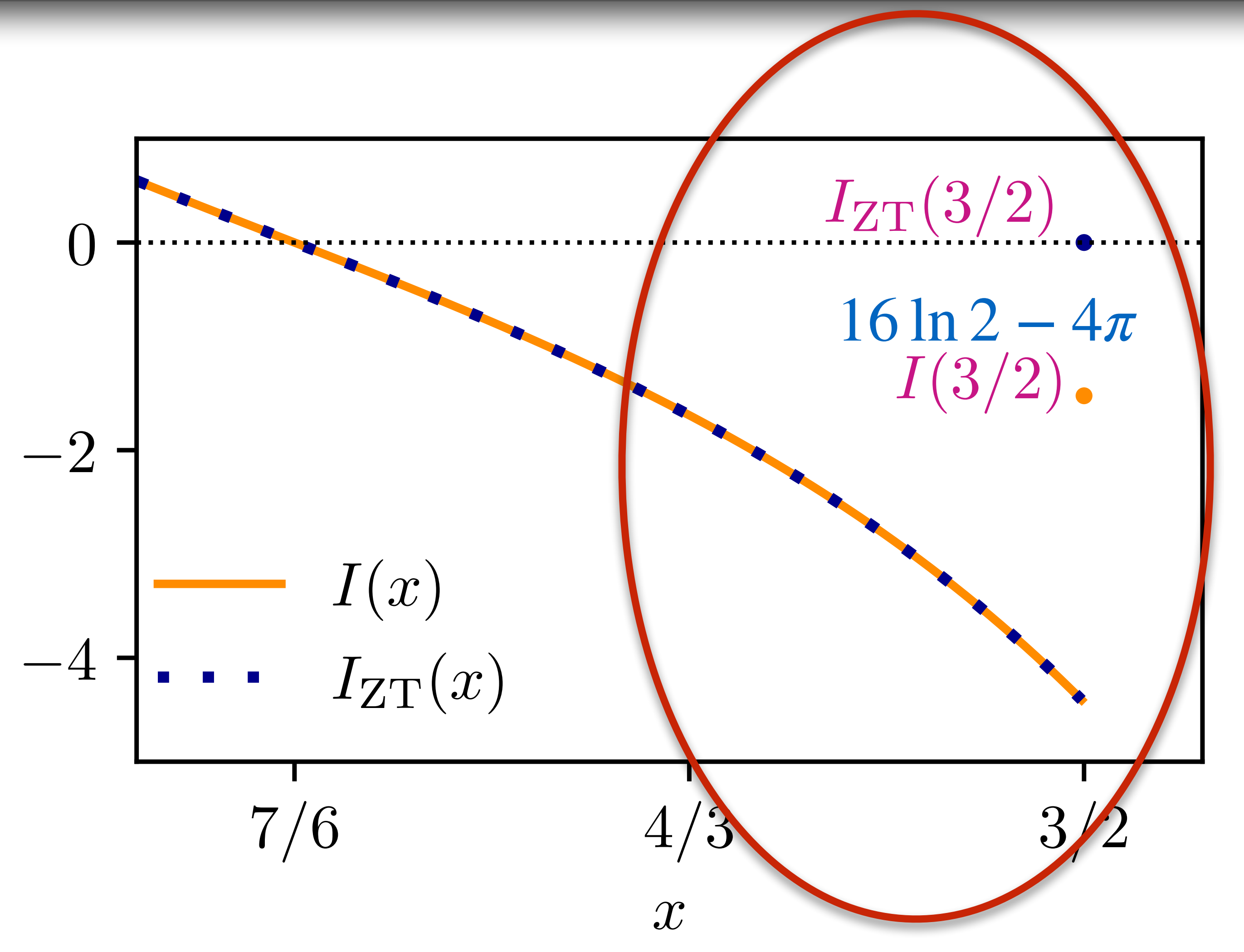
Fourier transform

3D GPE numerical simulations, spatio-temporal density spectra





Steady direct energy cascade: log-correction



$n_k = Ak^{-2x}$, $St_k = 4\pi^3 A^3 k^{4-6x} I(x)$,

$n_k = Ak^{-2x_P} = Ak^{-3}$

dimension analysis

$$P_k = \int_0^k 16\pi^4 A^3 \kappa^{8-6x} I(x) d\kappa$$

logarithmically divergent for $n_k \sim k^{-3}$, fake solution!!!

Phenomenologically, one can heal the divergence with a IR cut-off k_f and log-correction: $n_k \propto k^{-3} \log^{-1/3}(k/k_f)$

Dyachenko, et al. Physica D 57 (1992)
Kraichnan (2D enstrophy cascade)

Analytically, we find for $k \gg k_f$

Direct energy cascade KZ spectrum

$$n_k = C_d |P_O|^{1/3} k^{-3} \log^{-1/3}(k/k_f)$$

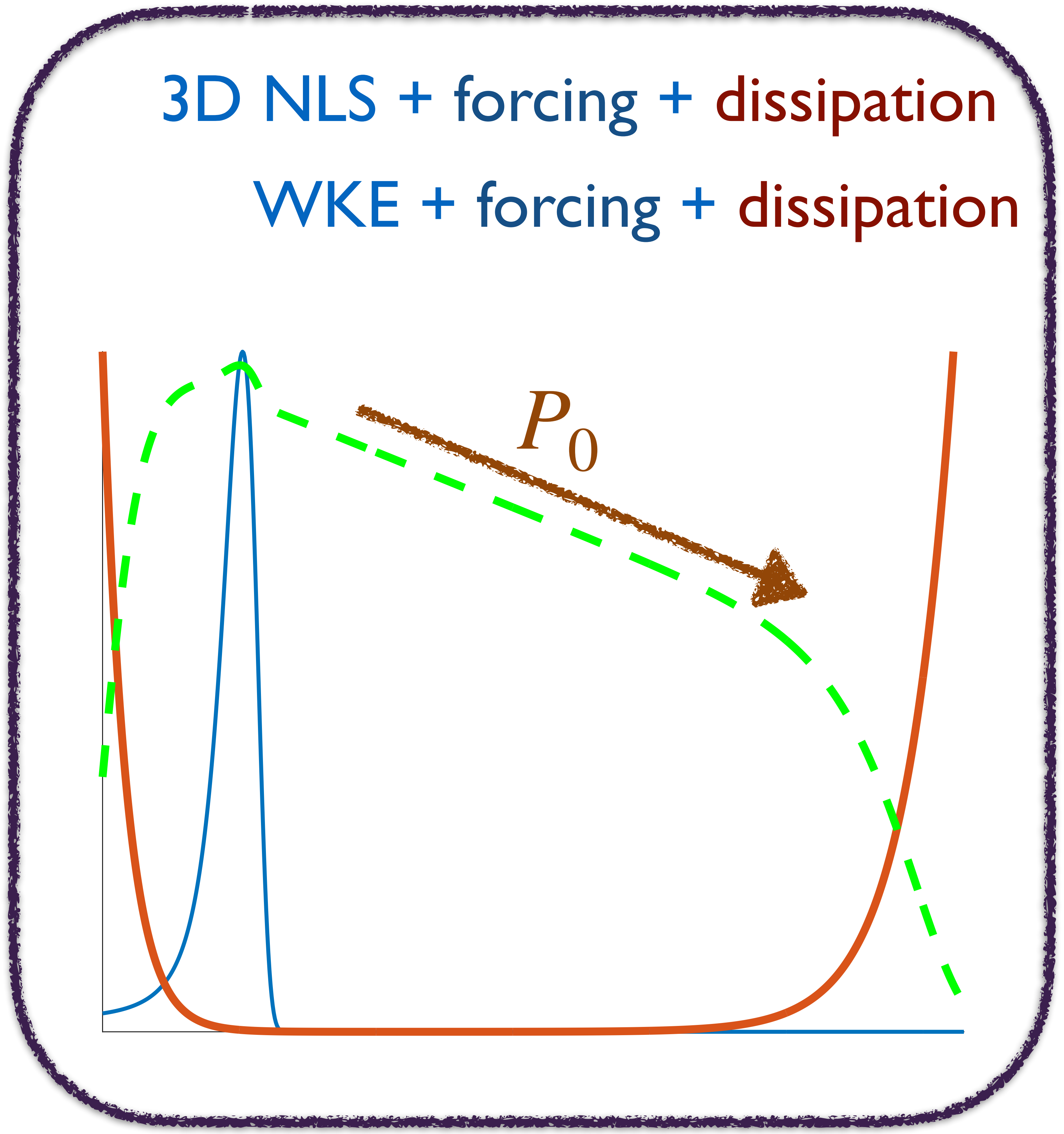
With $C_d \approx 5.26 \times 10^{-2}$ a universal constant

Steady direct cascade: numerical simulations

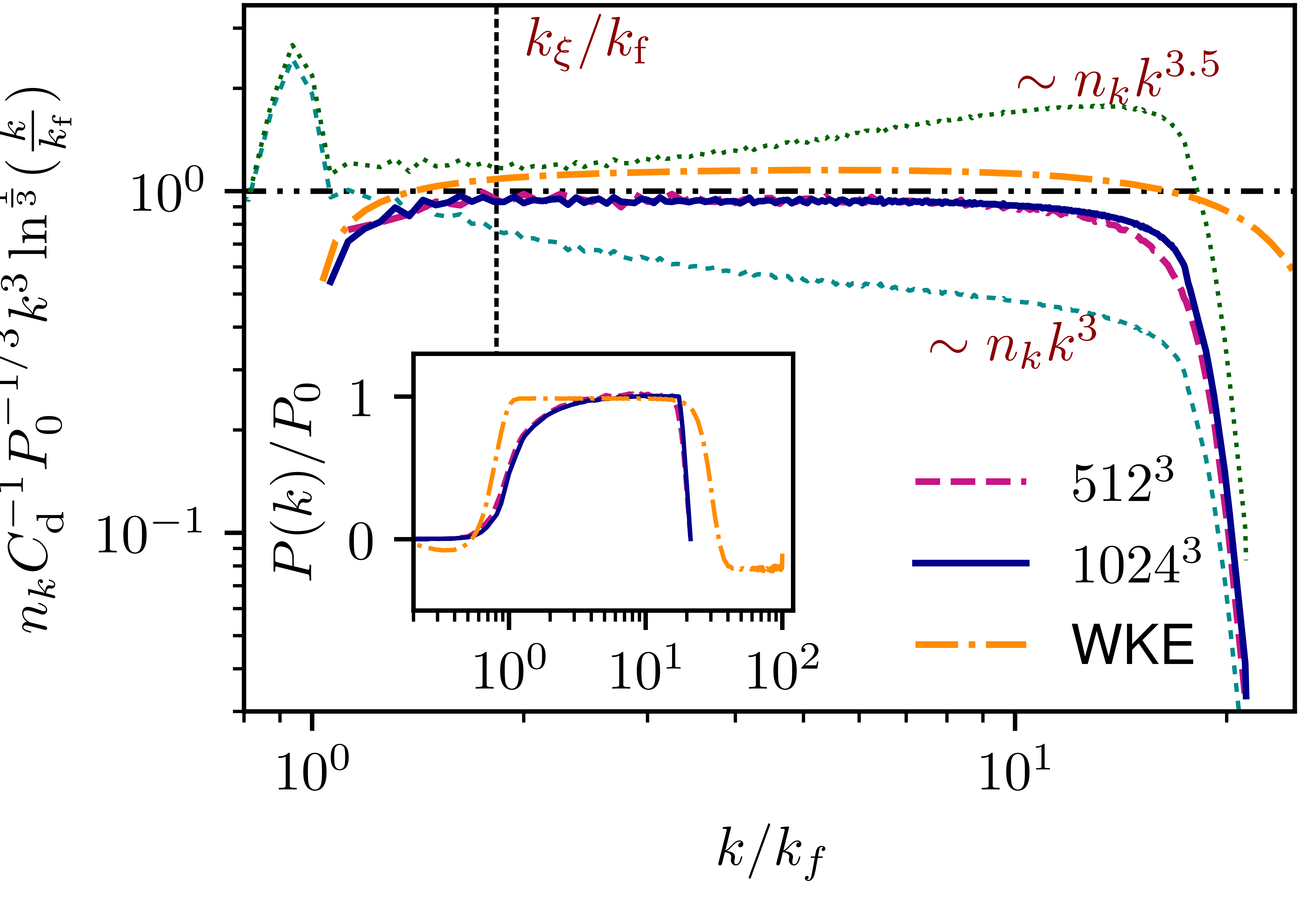
Direct energy cascade KZ spectrum

$$n_k = C_\mathrm{d}|P_O|^{1/3}k^{-3}\log^{-1/3}(k/k_\mathrm{f})$$

With $C_\mathrm{d} \approx 5.26 \times 10^{-2}$ a universal constant

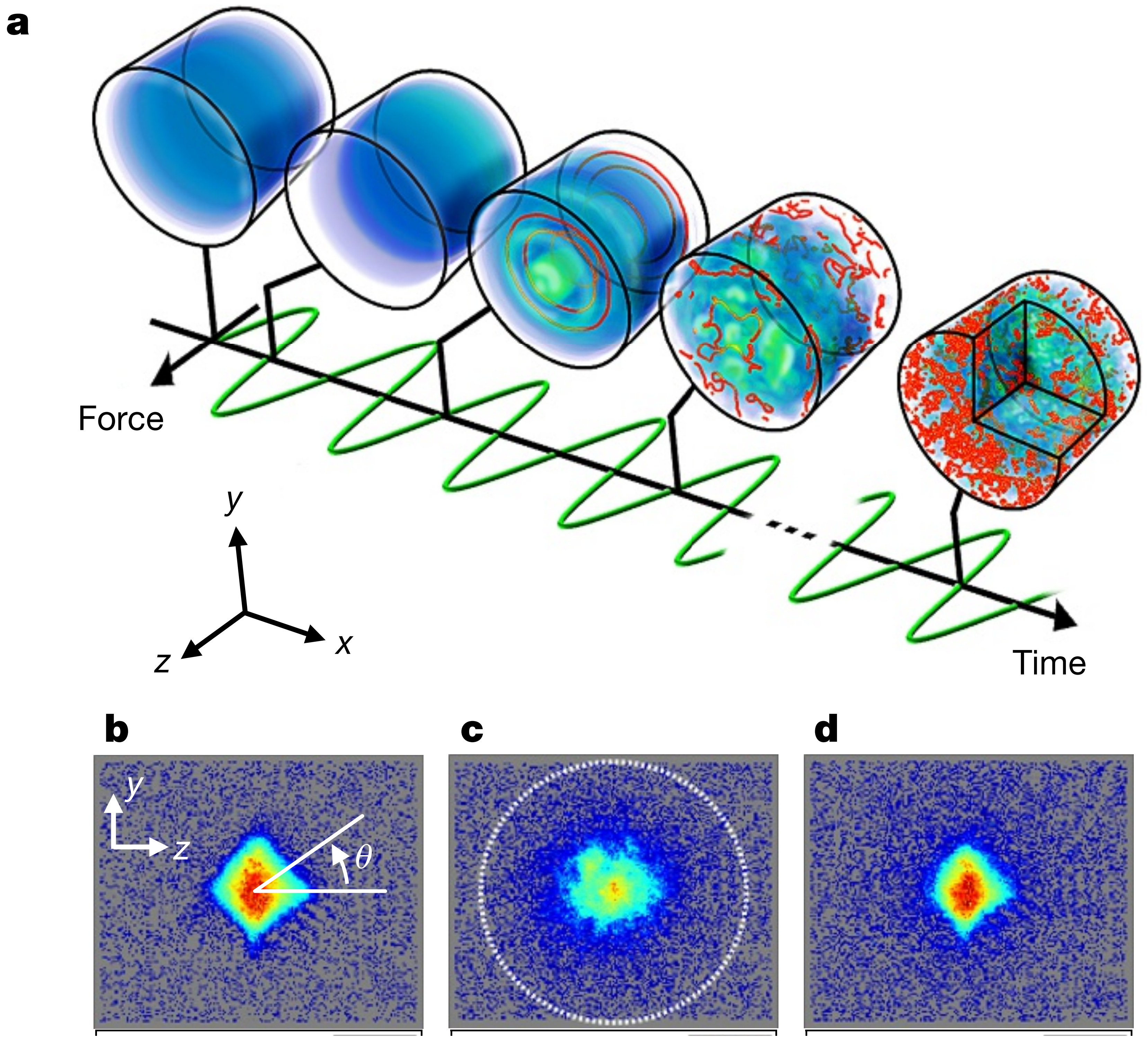


Numerical simulations of forced and dissipated 3D GPE and WKE

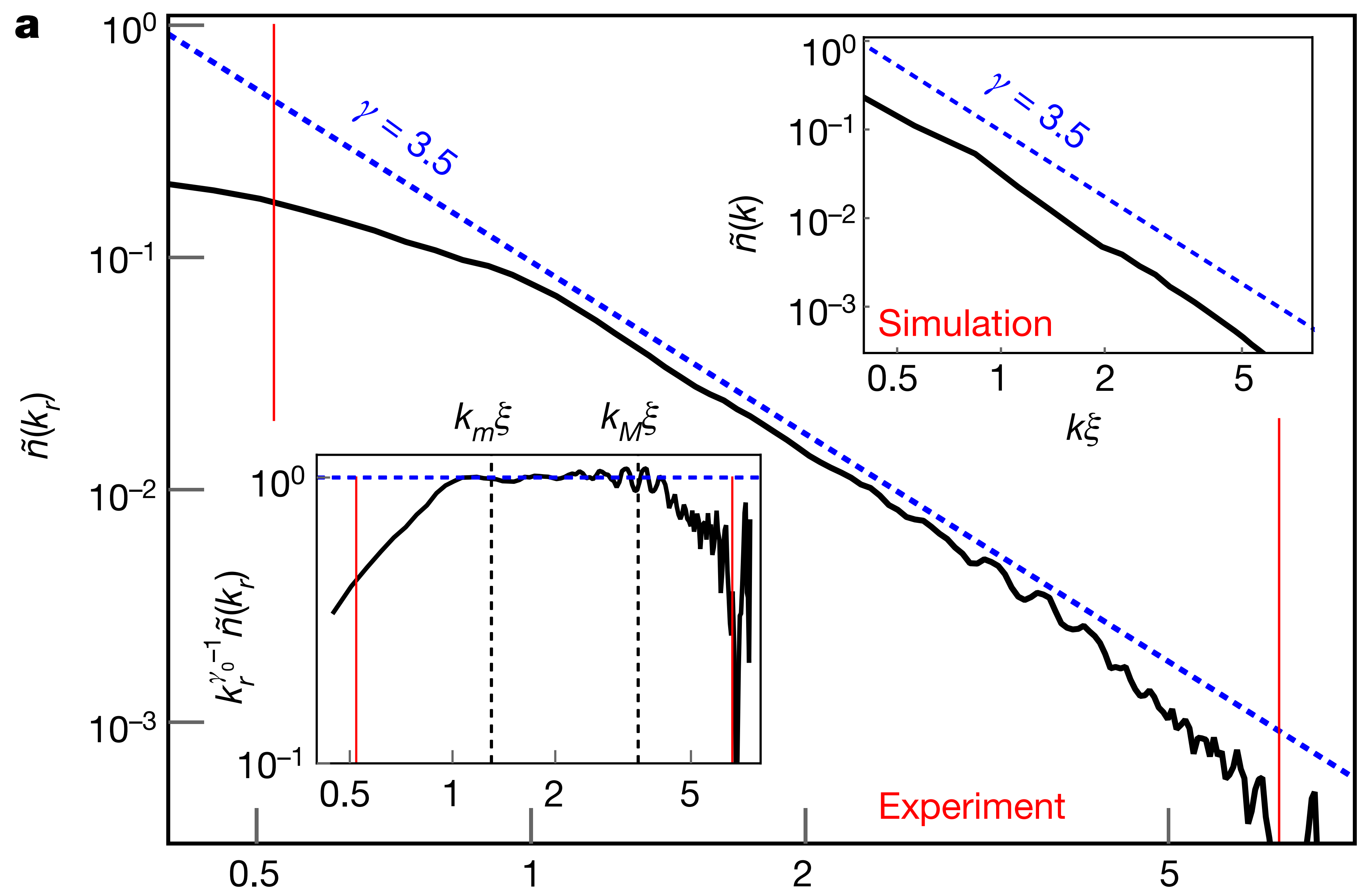


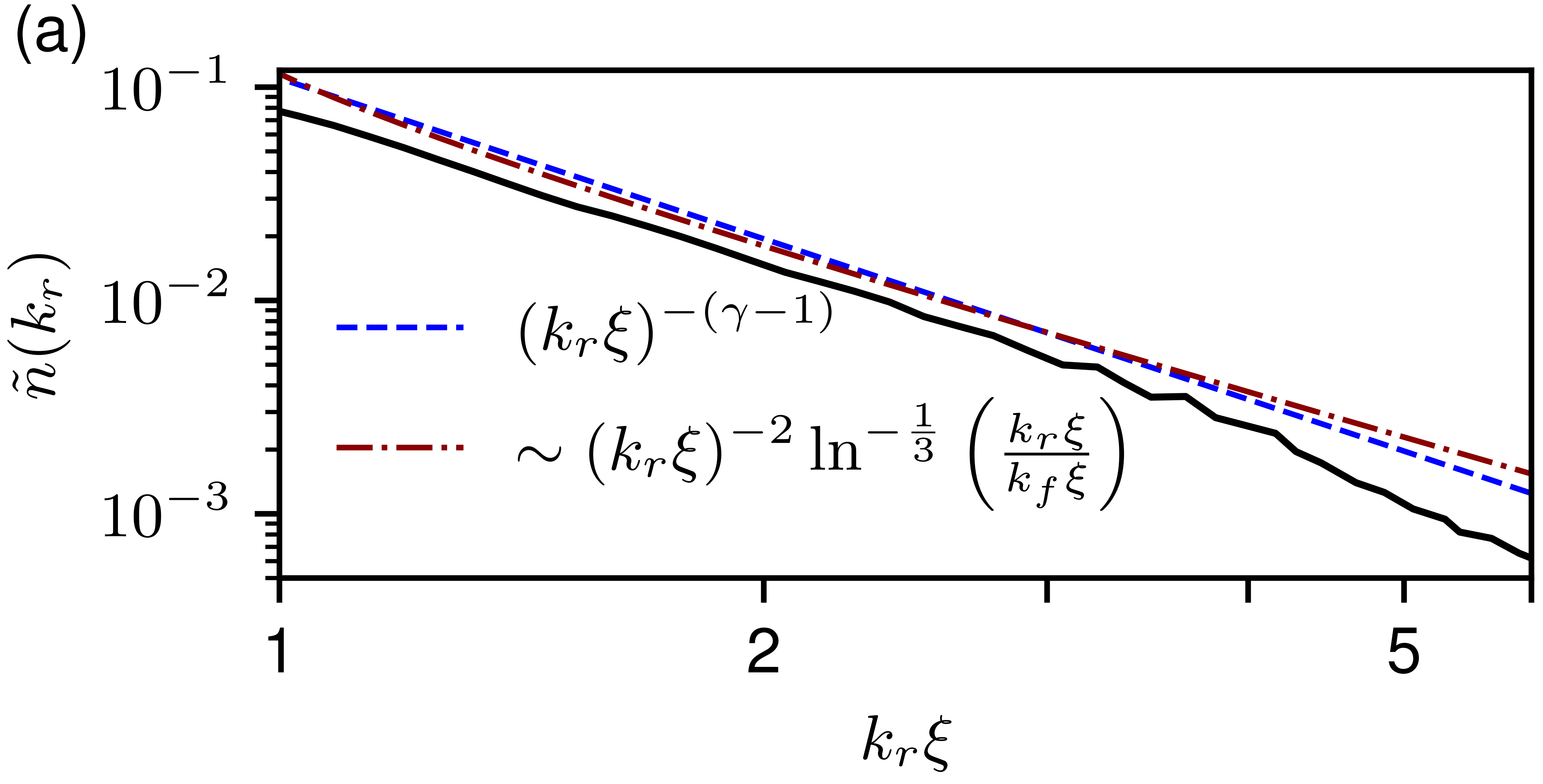
Steady direct cascade: experiments

Shaking a condensate in a $3D$ box



Navon et al. Nature 539, 72–75 (2016)





Steady direct cascade: experiments

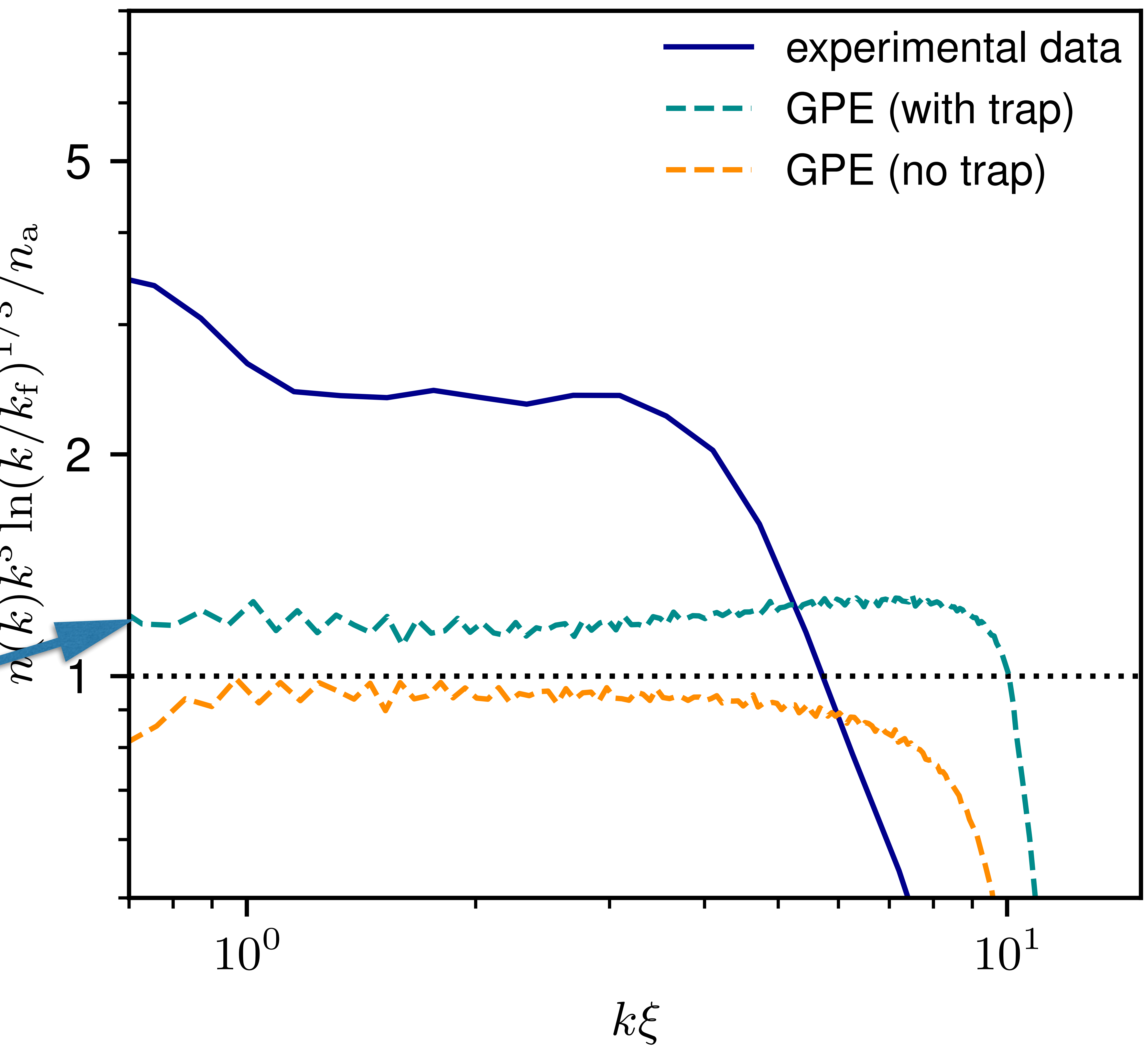
dimensional KZ solution for direct cascade

$$n_k = n_a k^{-3} \log^{-1/3}(k/k_{\rm f})$$

Equation of state $n_a = C_4 \left(\frac{\epsilon m^2}{\hbar^3 a^2} \right)^{1/3}$

$$N = V \int 4\pi k^2 n_k \mathrm{d}k$$

$C_4 = C_{\rm d}/(16\pi^2)^{1/3}$



$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U(\mathbf{x})\psi + g|\psi|^2 \psi$ + forcing + dissipation

Dogra, L.H., Martirosyan, G., Hilker, T.A. *et al.* *Nature* **620**, 521–524 (2023).

First-kind self-similarity in the direct range

Isotropic WKE for the radial $n_k^{\rm rad} = 4\pi k^2 n_k = k\, n_{2D}(k)$

$$\frac{\partial n_k^{\rm rad}}{\partial t} = 2\pi \int \frac{\min\left(k, k_1, k_2, k_3\right)}{k\, k_1 k_2 k_3} n_k^{\rm rad} n_{k_1}^{\rm rad} n_{k_2}^{\rm rad} n_{k_3}^{\rm rad} \delta(\omega_{23}^{01}) \left(\frac{k^2}{n_k^{\rm rad}} + \frac{k_1^2}{n_{k_1}^{\rm rad}} - \frac{k_2^2}{n_{k_2}^{\rm rad}} - \frac{k_3^2}{n_{k_3}^{\rm rad}} \right) \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3$$

Direct cascade's capacity is infinite $E = 4\pi \int_{k_f}^{\infty} k^2 \omega_k k^{-3} \ln^{-1/3}(k/k_f)\, dk = \infty$

self-similar solution of the first kind $n^{rad}(k,t) = t^{-1/2} f(\eta)$ **with** $\eta = k/t^b$

$$b = \lambda/3 + 1/6\,, \quad \text{if} \quad E(t) \sim t^{\lambda}$$

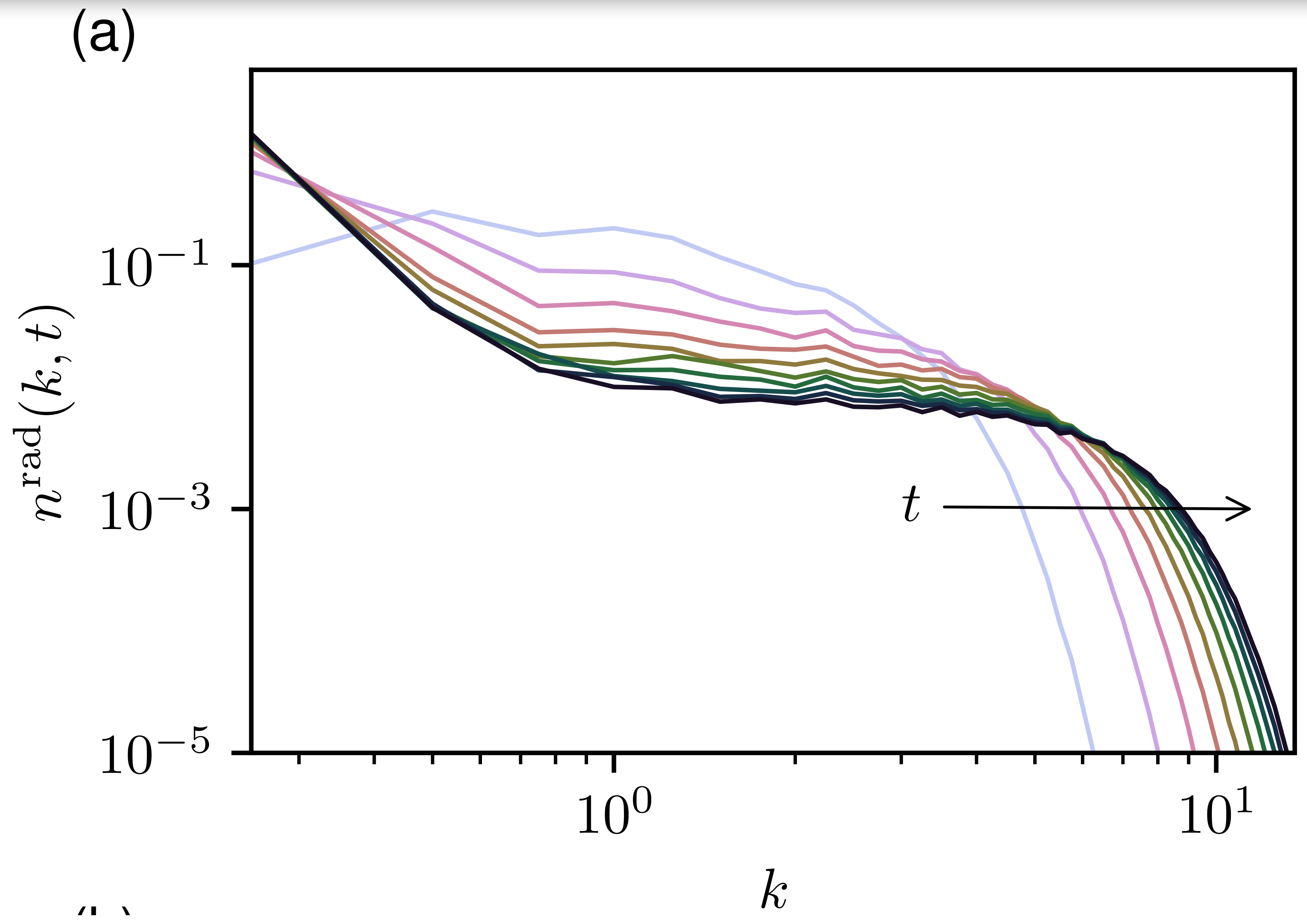
Convert to $n_k(t) = n(k,t)$ $\quad n_k(t) = t^{-1/2-2b} \tilde{f}(\eta)$ **with** $\eta = k/t^b$

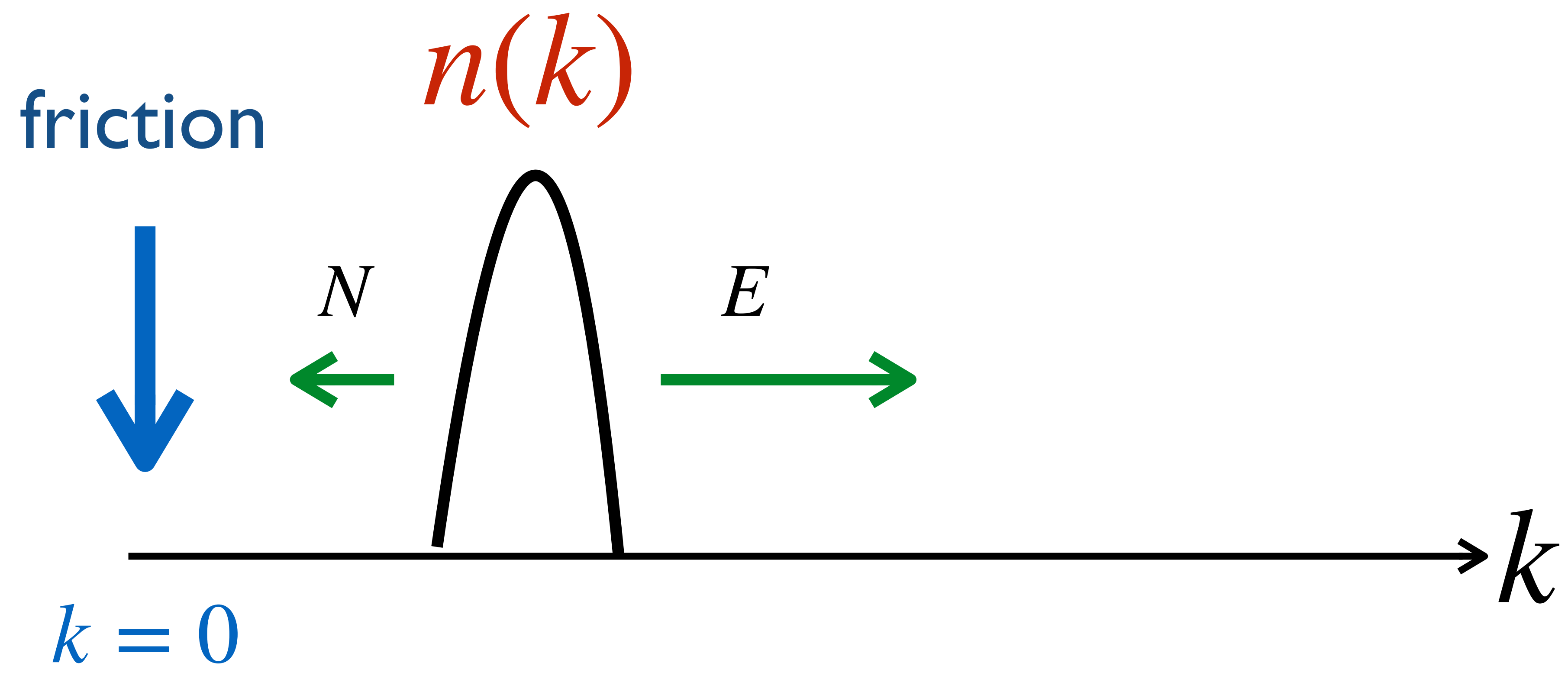
Convert to $n_{2D}(k,t)$ $\quad n_{\rm 2D}(k,t) = t^{-1/2-b} \hat{f}(\eta)$ **with** $\eta = k/t^b$

- Free system: $E = const \longrightarrow b = 1/6$
- Forced system: $E \sim t \longrightarrow b = 1/2$

Stationary spectra in the wake: RJ for the free case, KZ for the forced case.

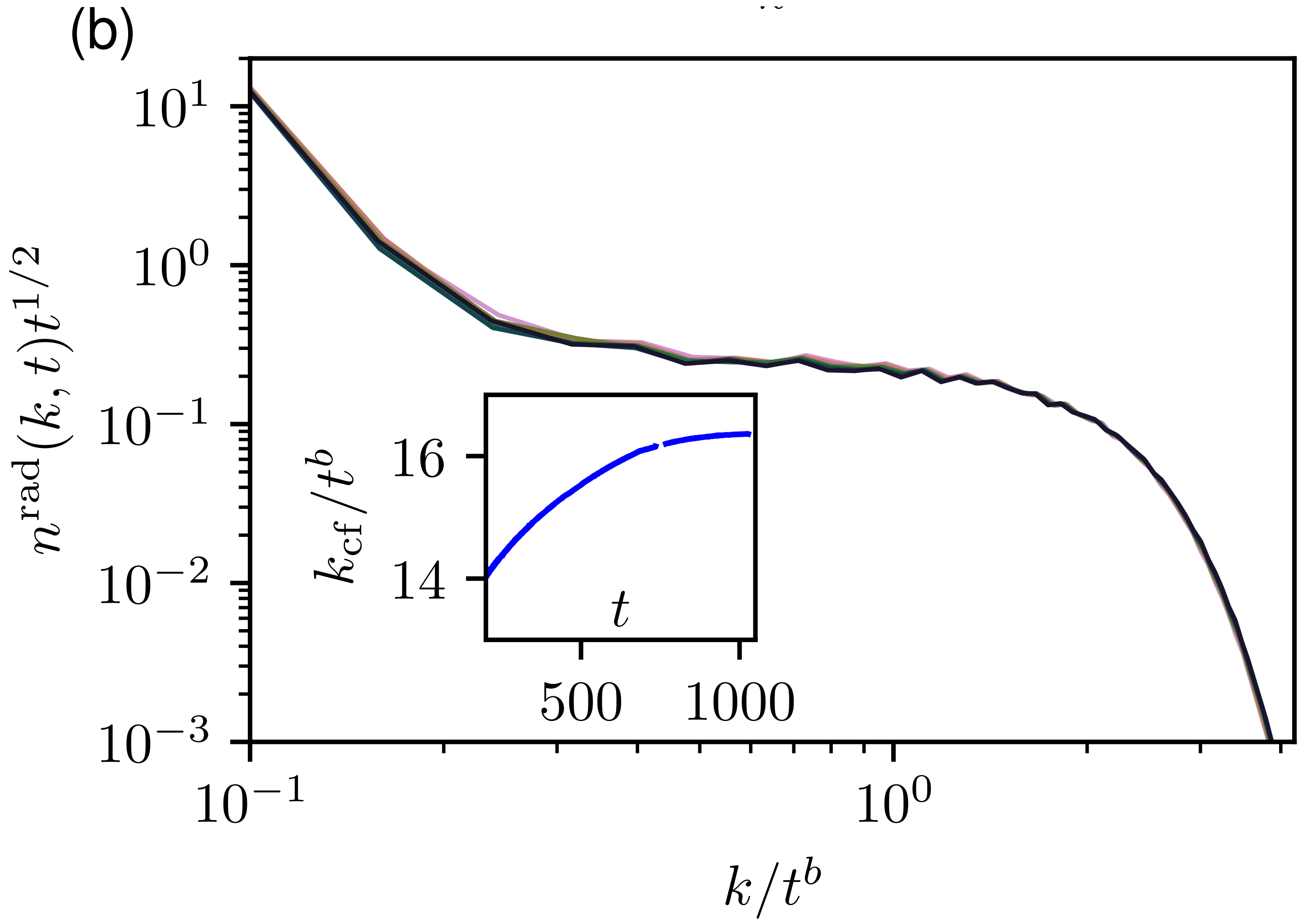
Free direct cascade evolution (GPE)





- Spectrum behind the front is RJ
- Perfect collapse with the predicted self-similar shape

In self-similar regime, wavefront $k_{\rm cf}/t^b \to$ const



Theoretical prediction: $n^{rad}(k,t) = t^{-a} f(k\,t^{-b})\,, a = 1/2\,, b = 1/6$

Experimental observations

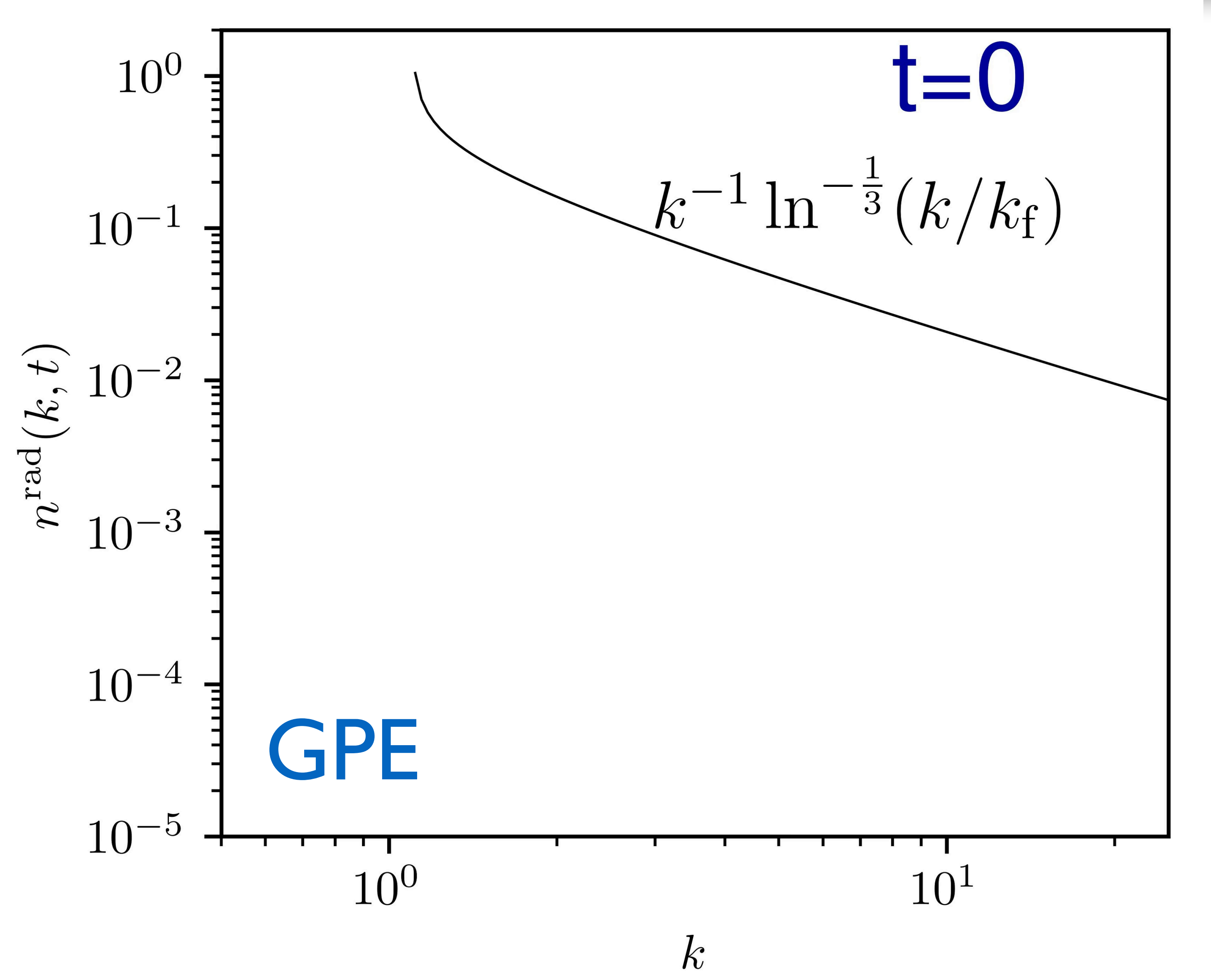
Glidden J A P, Eigen C, Dogra L H, et al. Nature Physics, 2021, 17(4): 457-461.

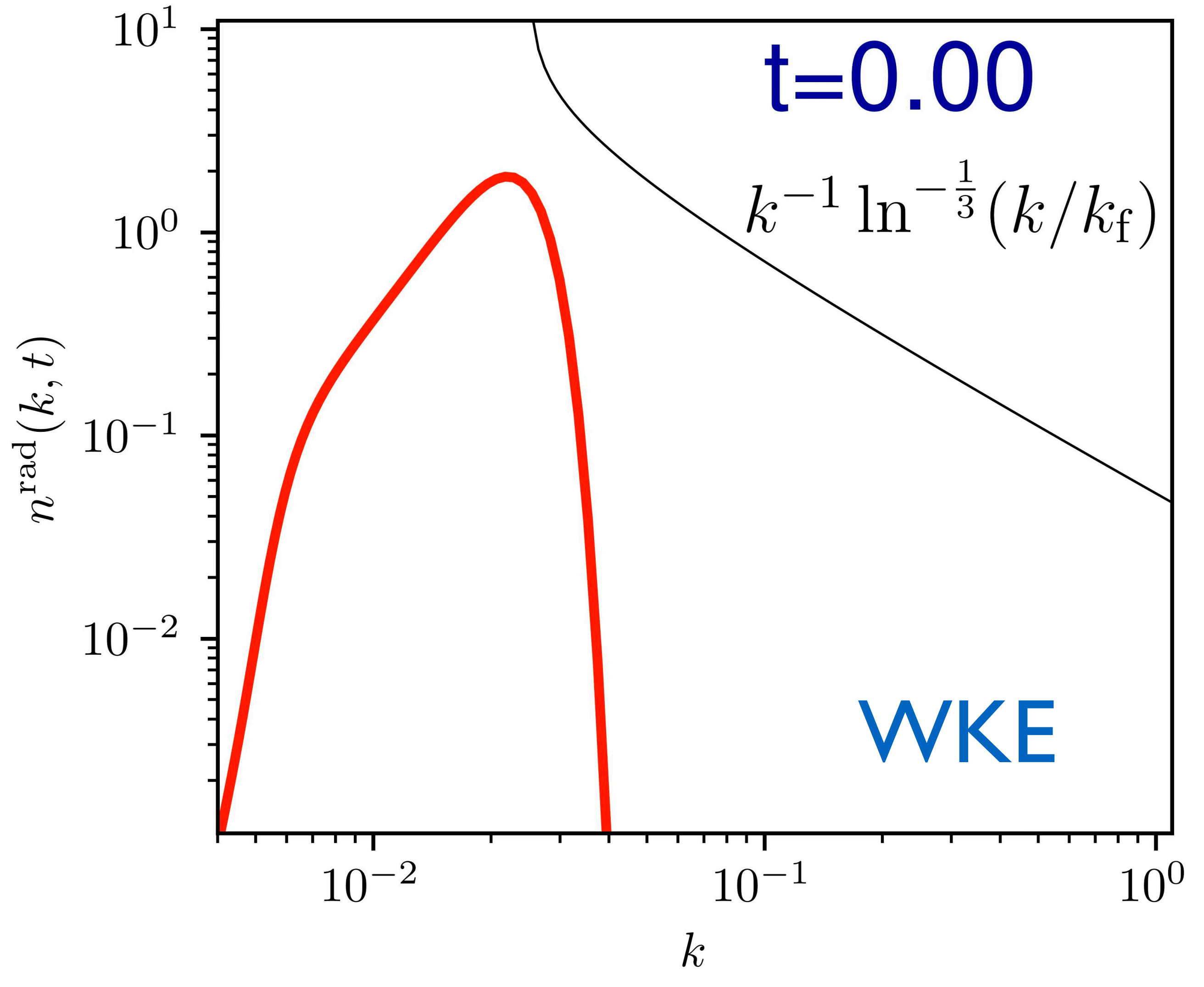
$a = 0.4(1)\,,\ b = 0.14(2)$

García-Orozco A D, Madeira L, Moreno-Armijos M A, et al. Physical Review A, 2022, 106(2): 023314.

$a = 0.3(1)\,,\ b = 0.2(4)\,,\ n(k) \sim k^{-2}$

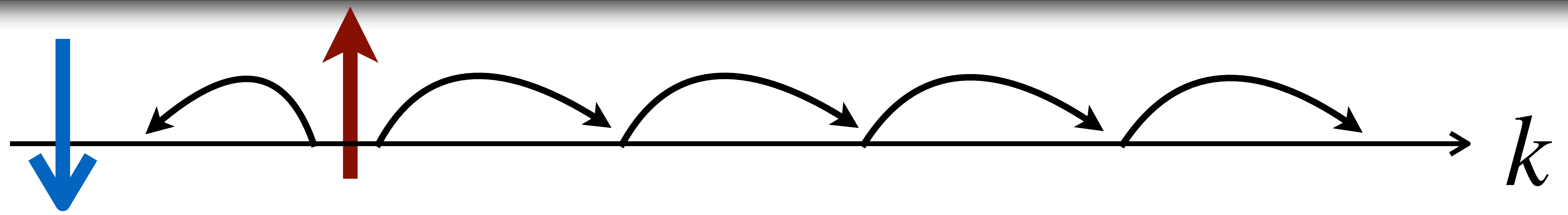
Forced direct cascade evolution

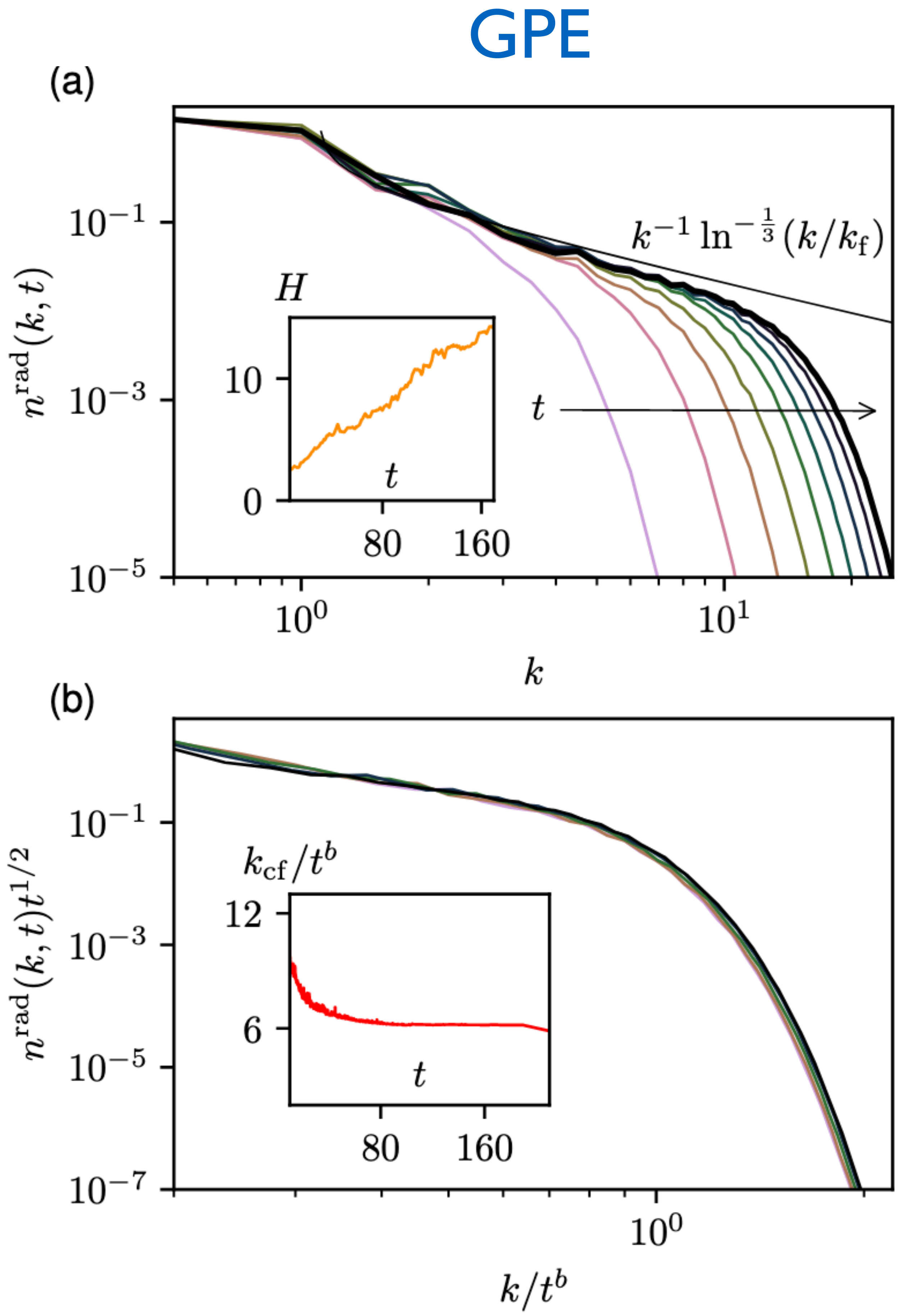


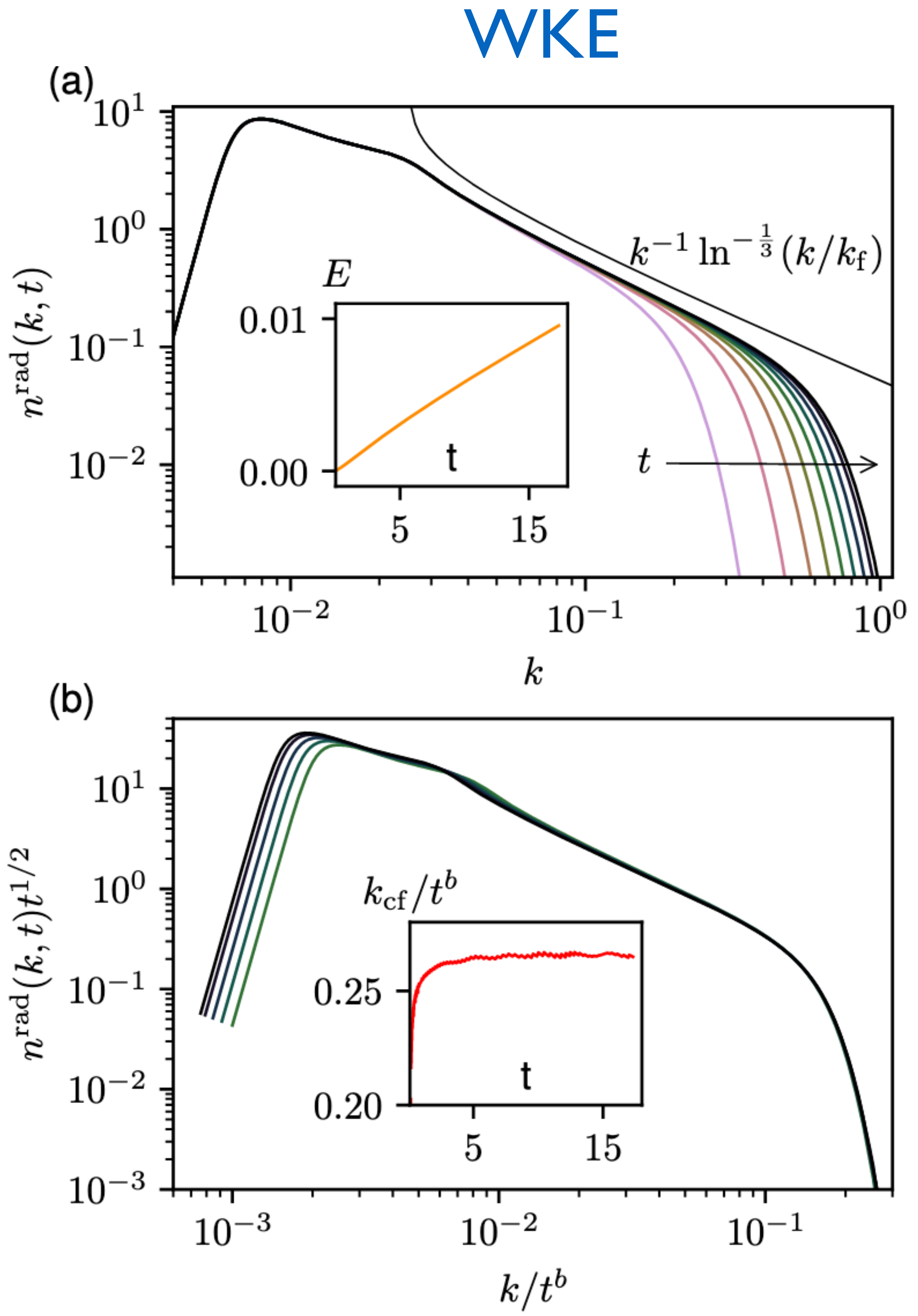


Relevant experiment

Navon, Science 366, (2019)







- Spectrum behind the front is direct-cascade KZ
- Perfect collapse with the predicted self-similar shape

Steady inverse particle cascade

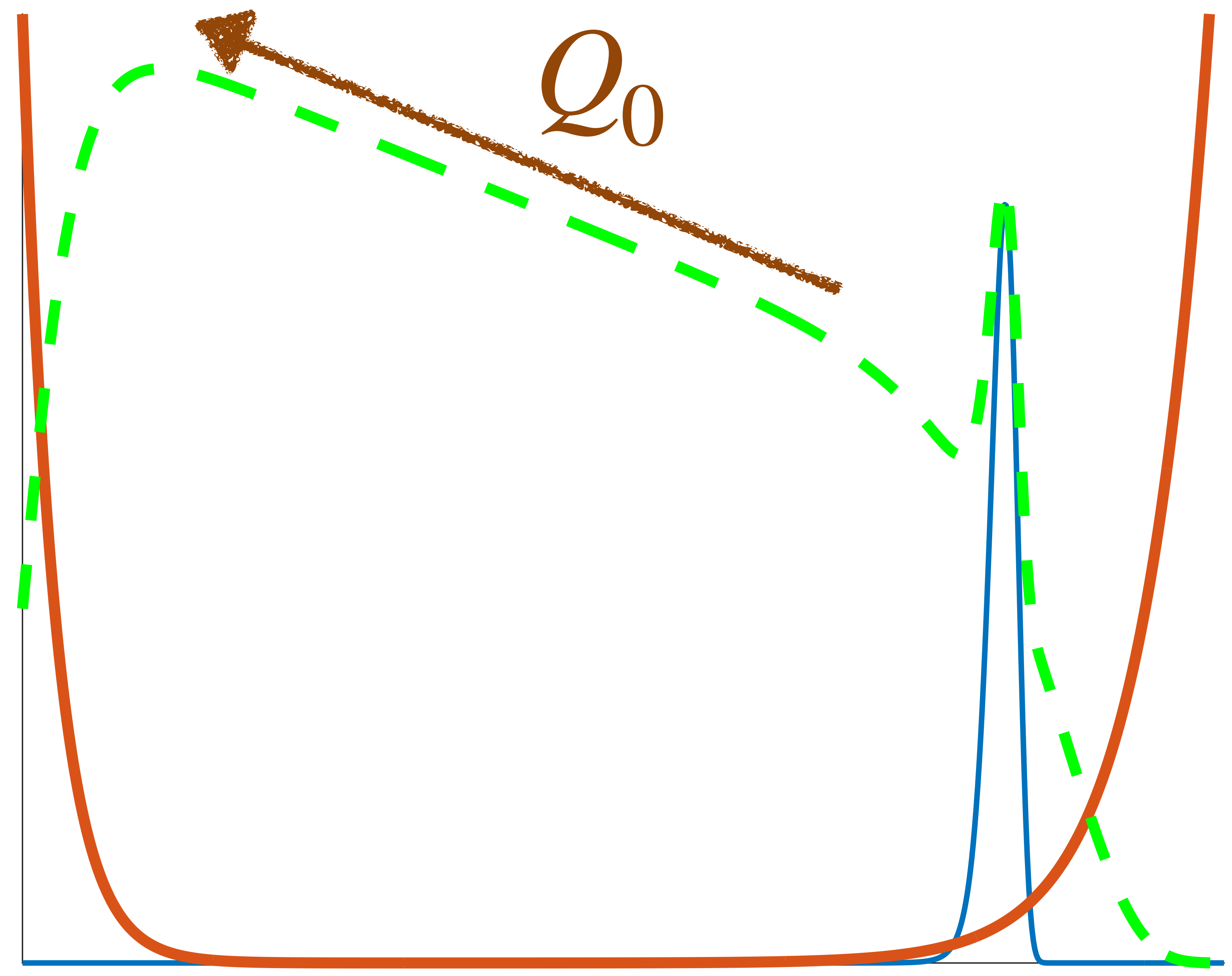
Inverse particle cascade KZ spectrum

$$n_k = C_{\mathrm{i}} \, |Q_O|^{1/3} \, k^{-7/3}$$

With $C_{\mathrm{i}} \approx 7.5774045 \times 10^{-2}$
a universal constant

$$C_{\mathrm{i}} = \frac{1}{2\pi^{3/2}} \Gamma\left(\tfrac{5}{6}\right)^{1/3} \left[3\Gamma\left(\tfrac{1}{3}\right) \left(3^{3/2}\, 2^{2/3}\, {}_3F_2\left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}; \tfrac{4}{3}, \tfrac{4}{3}; 1\right) - 8\, {}_3F_2\left(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}; \tfrac{4}{3}, \tfrac{3}{2}; 1\right) + 2^{1/3}\, {}_3F_2\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{5}{3}; 1\right) - 2^{1/3}\, {}_4F_3\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{5}{3}; 1\right) \right) \right]^{-1/3} \approx 7.5774045 \times 10^{-2}$$

3D GPE + forcing + dissipation
WKE + forcing + dissipation

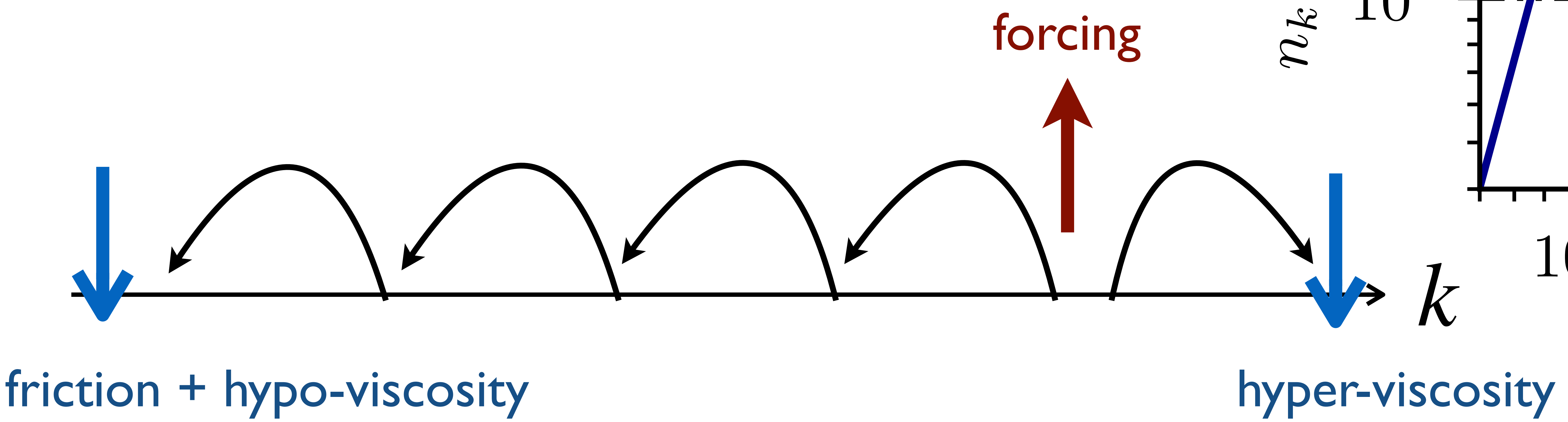


Steady inverse cascade: numerical simulations

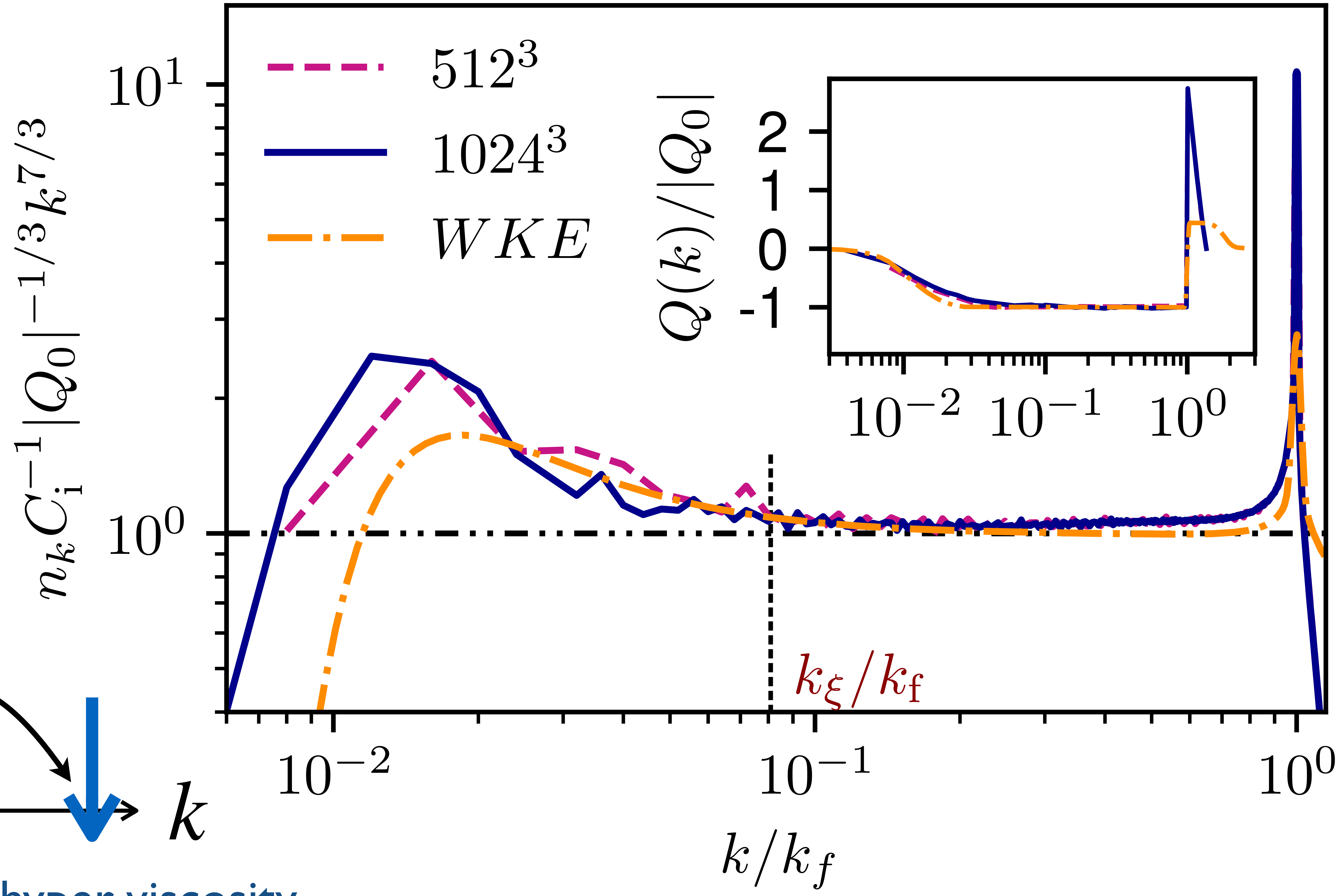
Inverse particle cascade KZ spectrum

$$n_k = C_{\mathrm{i}} |Q_O|^{1/3} k^{-7/3}$$

With $C_{\mathrm{i}} \approx 7.5774045 \times 10^{-2}$
a universal constant



Numerical simulations of forced and dissipated 3D GPE and WKE



Second-kind self-similarity in the inverse range

$$\frac{\partial n_k^{\rm rad}}{\partial t} = 2\pi \int \frac{\min(k, k_1, k_2, k_3)}{k\, k_1 k_2 k_3} n_k^{\rm rad} n_{k_1}^{\rm rad} n_{k_2}^{\rm rad} n_{k_3}^{\rm rad}\, \delta(\omega_{23}^{01}) \left(\frac{k^2}{n_k^{\rm rad}} + \frac{k_1^2}{n_{k_1}^{\rm rad}} - \frac{k_2^2}{n_{k_2}^{\rm rad}} - \frac{k_3^2}{n_{k_3}^{\rm rad}} \right) {\rm d}k_1 {\rm d}k_2 {\rm d}k_3$$

Inverse cascade's capacity is finite $N = 4\pi \int_0^{k_{\max}} k^2 k^{-7/3}\, dk < \infty$

self-similar solution of the second kind $n^{rad}(k,t) = \tau^{-1/2} g(\eta)$ **with** $\eta = k/\tau^m, \tau = t^* - t$

Satisfying $f(\eta) \to \eta^2$ for $\eta \to 0$ and $f(\eta) \to \eta^{-x^*}$ for $\eta \to \infty$. ⟶ $x^* = 1/2m$

Candidates of $x^* = 0.5, 0.44, 0.48, 0.56$ $> 1/3$ (steady inverse KZ scaling)

Semikoz and Tkachev 1995, Lacaze et al 2001.

Semisalov et at 2021.

Shukla and SN 2020.

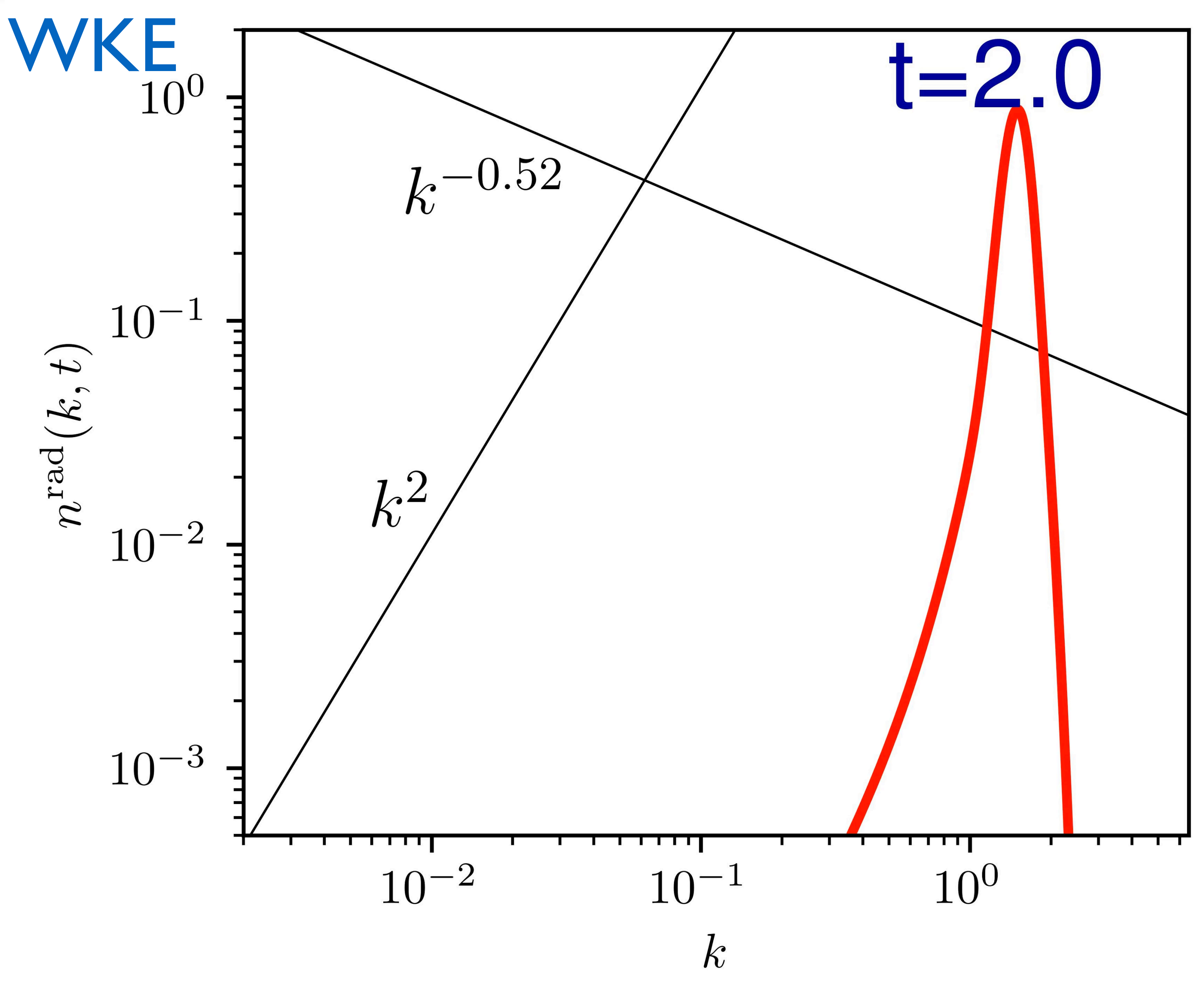
Moreno-Armijos M A et al 2004. $x^* = 0.6$

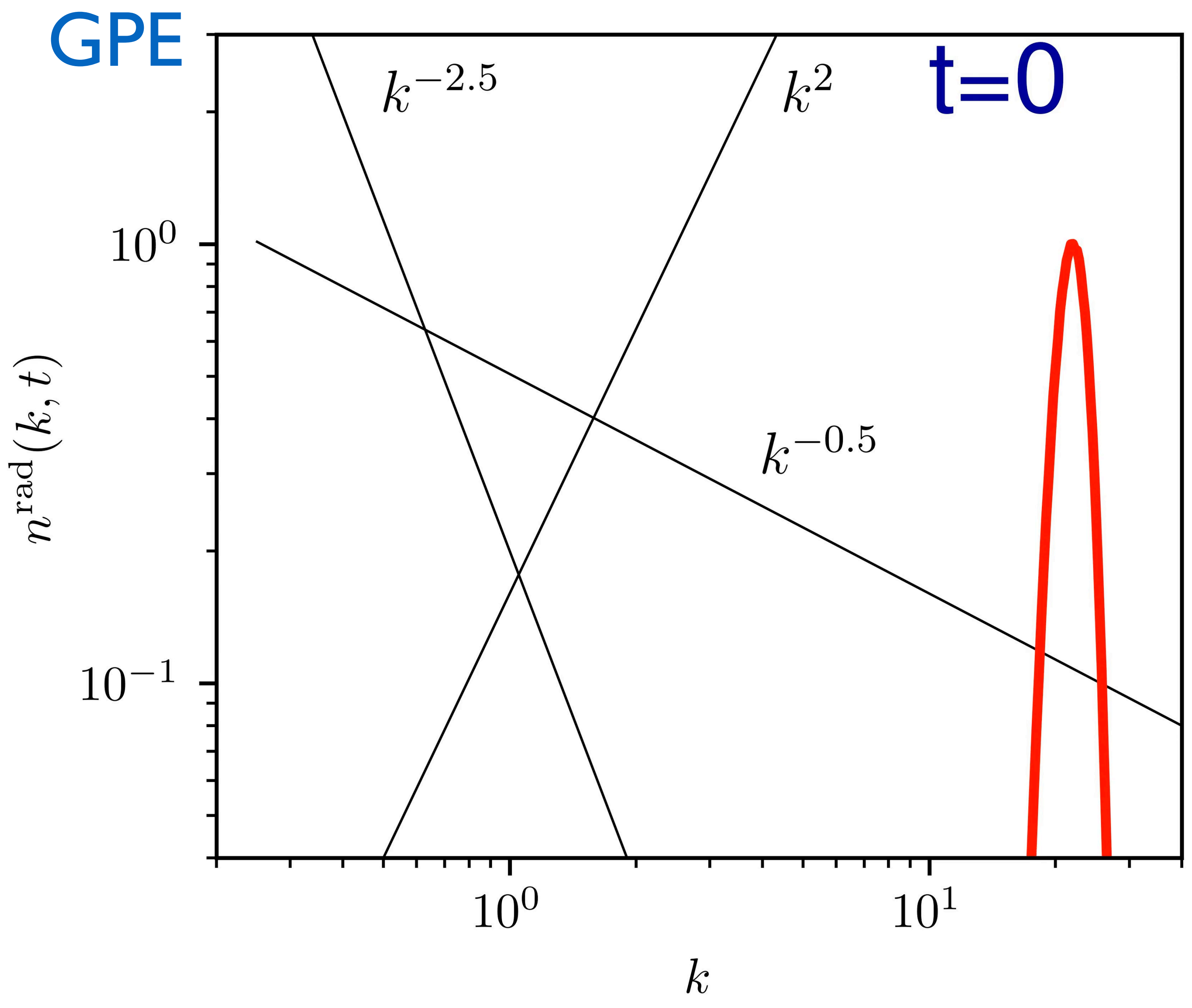
Take $x^* = 0.5$ $\quad n_k^{\rm rad}(t) = \tau^{-1/2} g(k/\tau), \quad n_k^{\rm rad}(t) \sim k^{-0.5}$

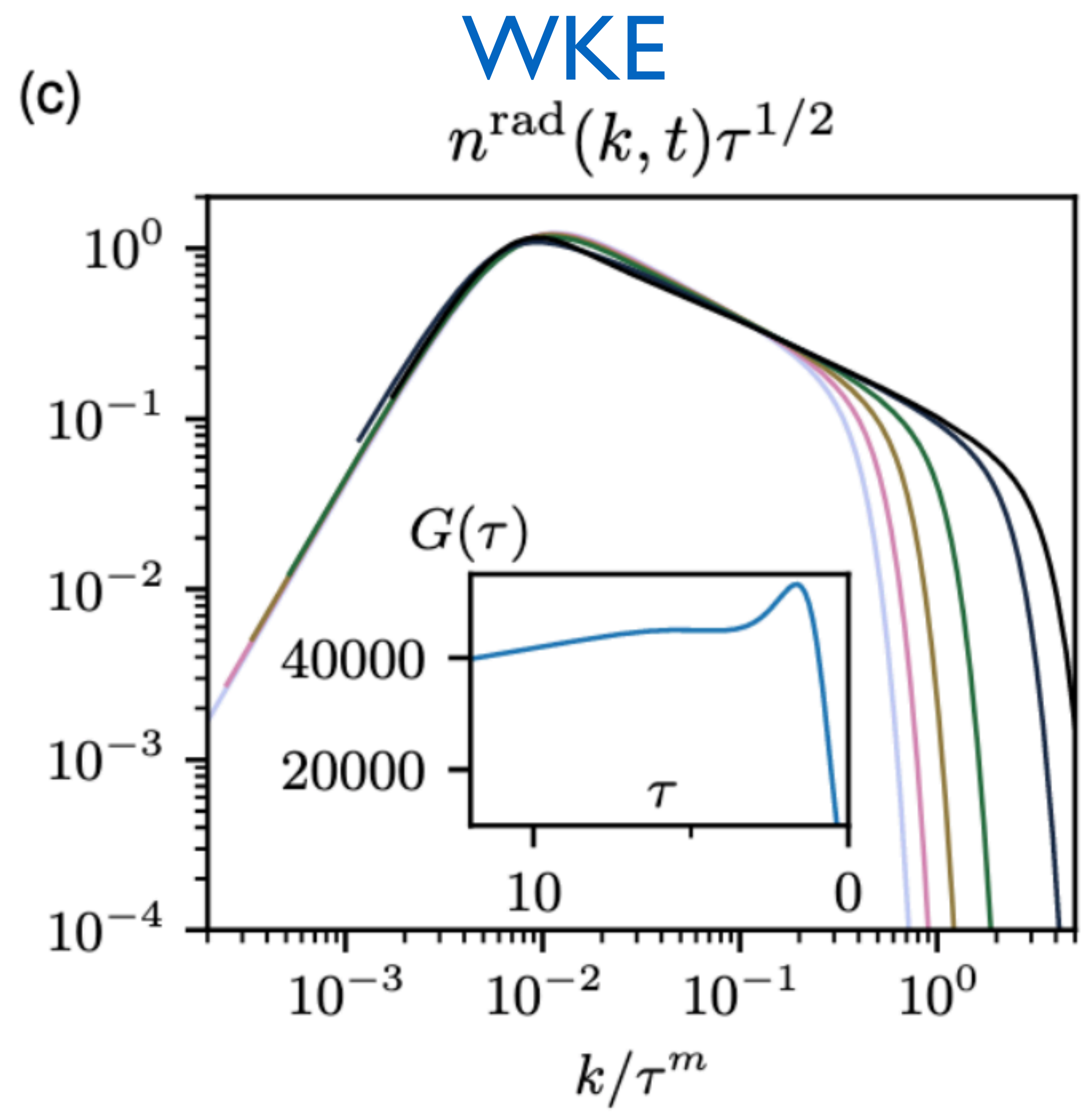
Convert to $n_k(t) = n(k,t)$ $\quad n_k(t) = \tau^{-2.5} \tilde{g}(k/\tau), \quad n_k \sim k^{-2.5}$

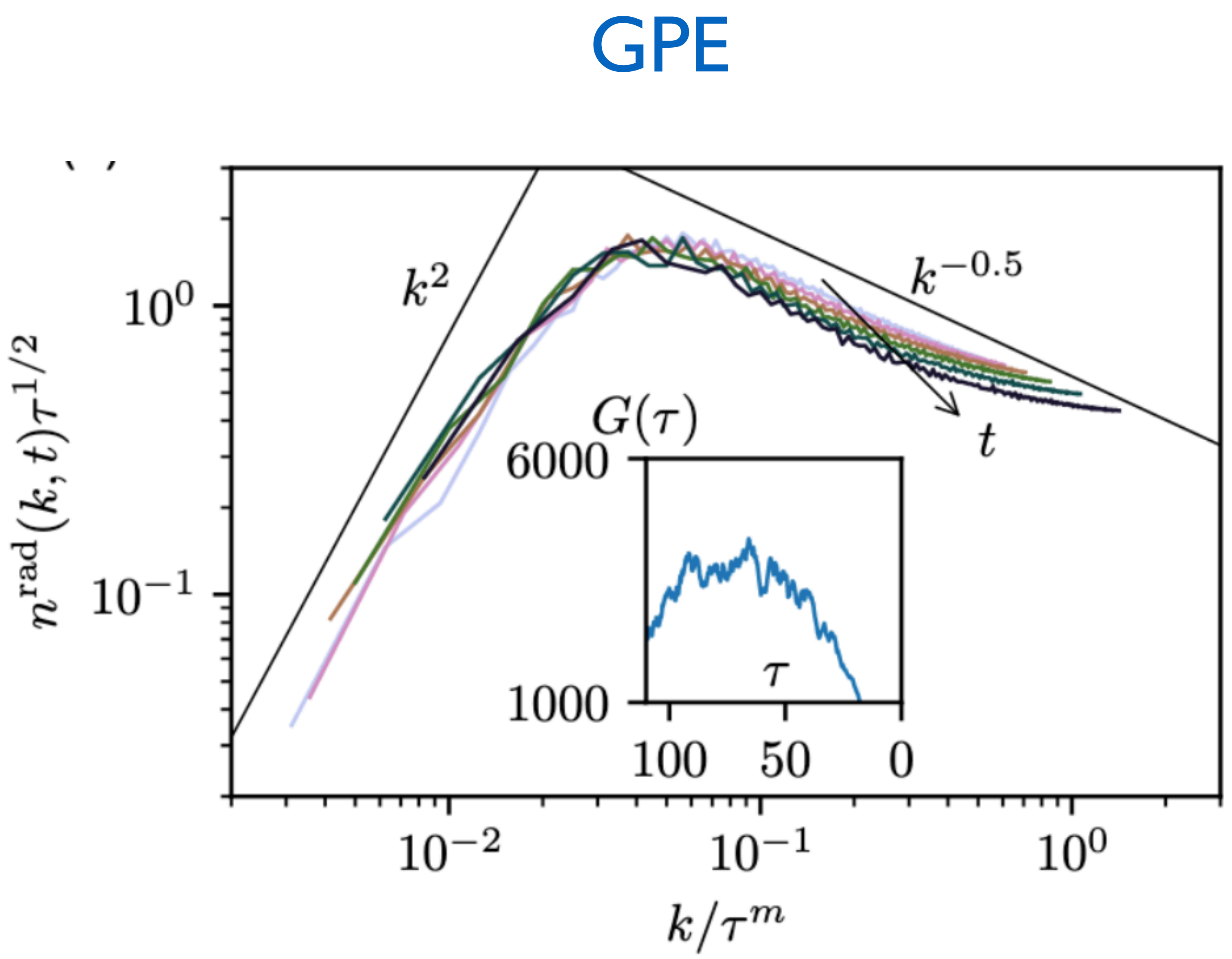
Convert to $n_{2D}(k,t)$ $\quad n_{\rm 2D}(k,t) = \tau^{-1.5} \hat{g}(k/\tau), \quad n_{\rm 2D}(k,t) \sim k^{-1.5}$

Free inverse cascade evolution





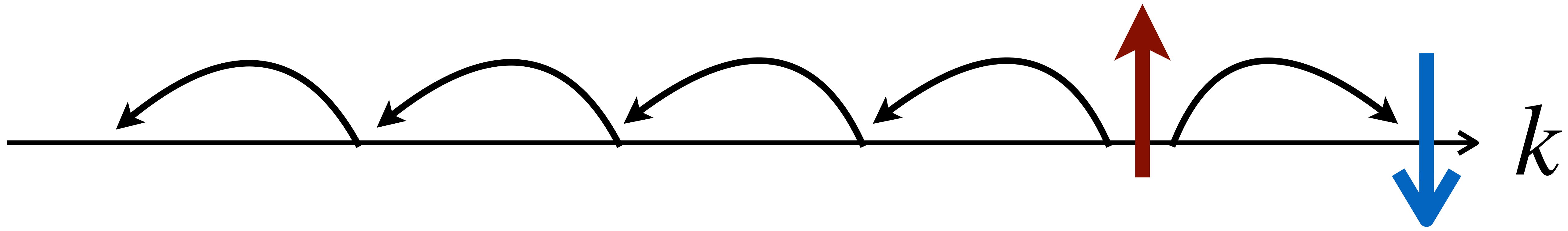




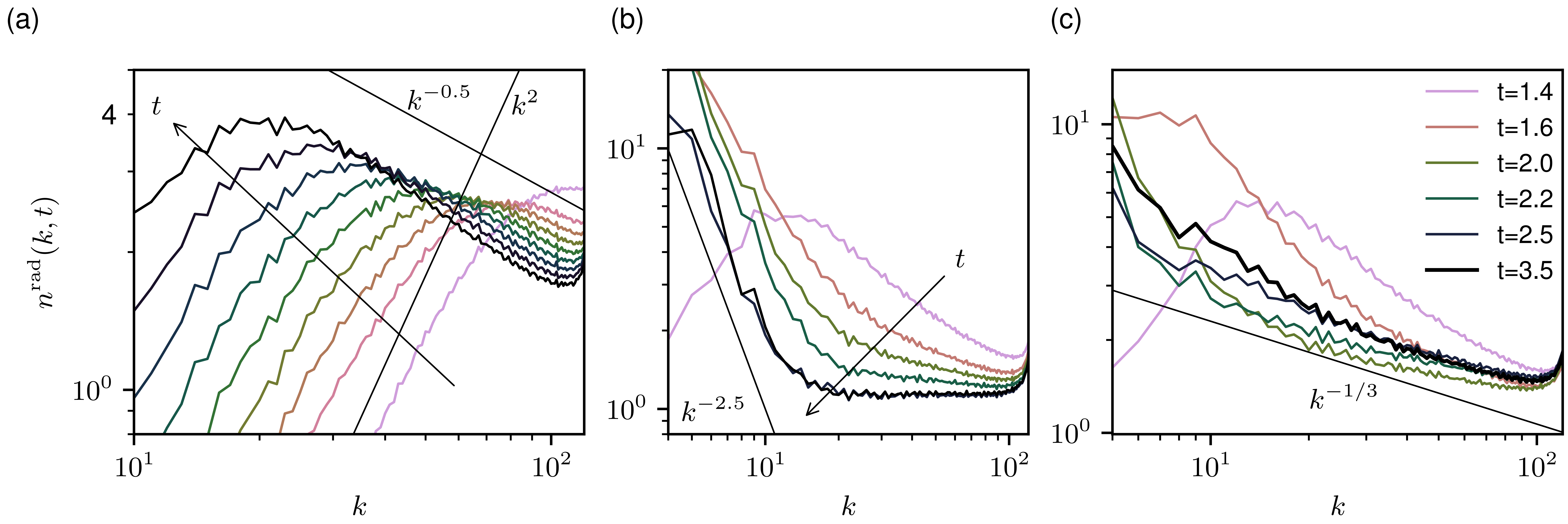
In self-similar regime, $G(\tau) = \lim_{k\to 0} n^{\rm rad}(k,t)\tau^{1/2+2m}$ → const as $\tau \to 0$

What do forced systems look like ?

Free and forced systems behave the same because of the "infinite" reservoir of N at high k.



Relaxation to steady inverse KZ solution (GPE)



Short-tem evolution

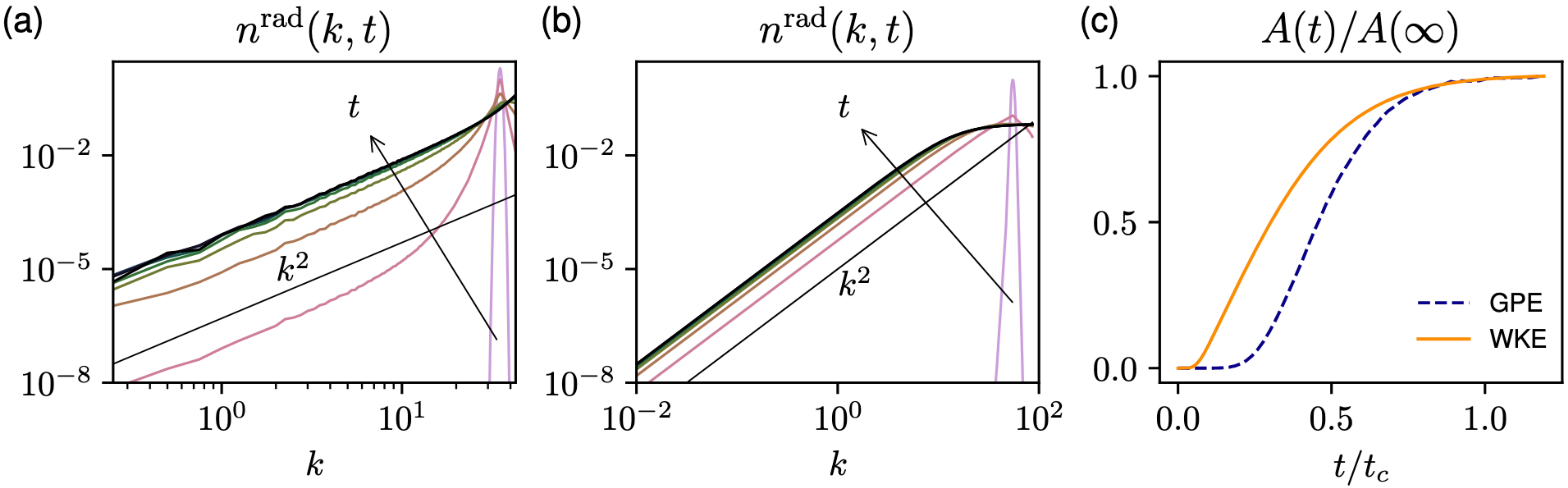
Long-term evolution without dissipation at low k

Long-term evolution with dissipation at low k

Blowup and Condensation

Blow-up condition: $E/N < (E/N)_c = \frac{k_{\max}^2}{3}$

For $E/N > (E/N)_c = \frac{k_{\max}^2}{3}$: no blowup!



GPE WKE

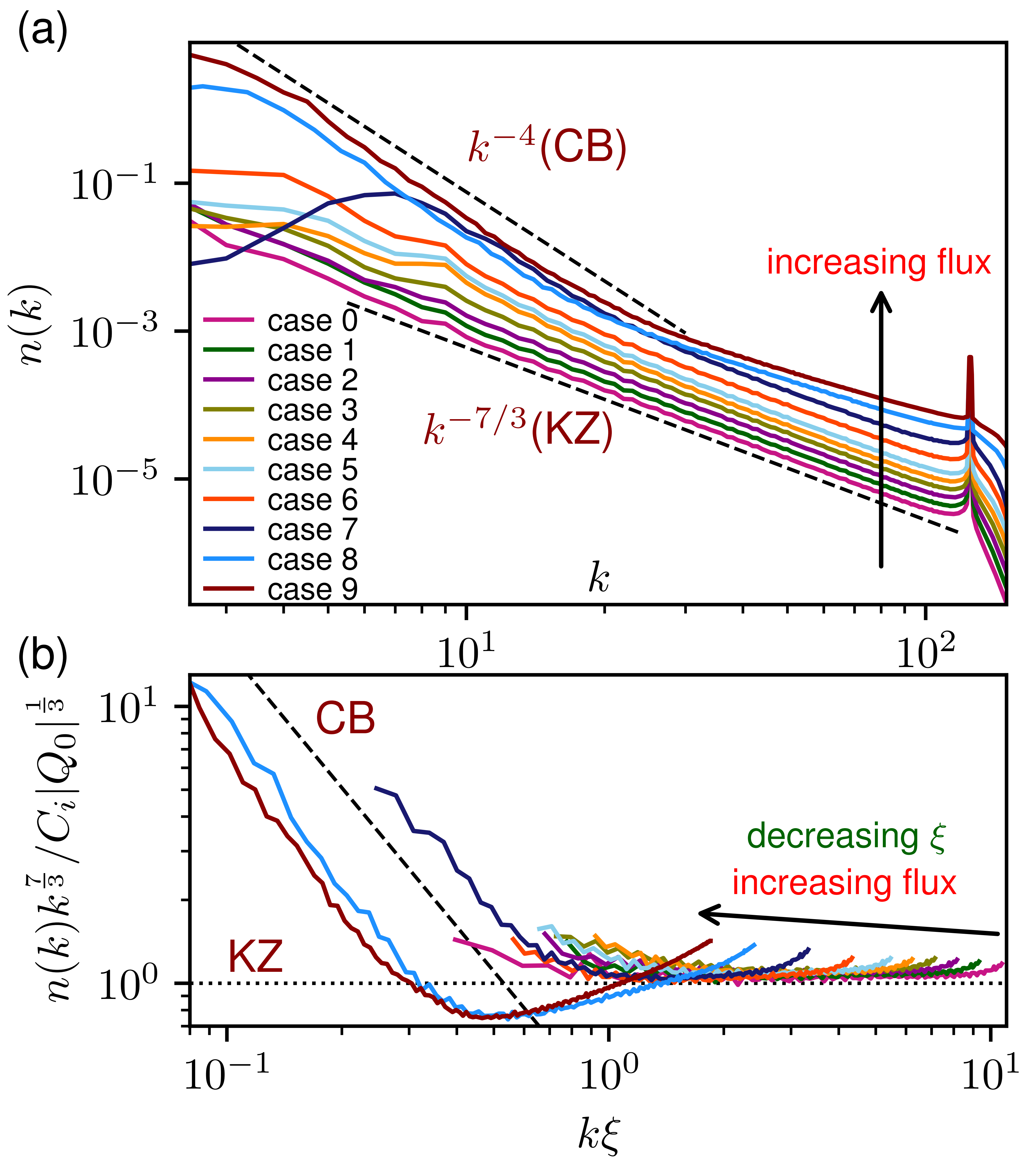
➡ Asymptote to a steady state RJ spectrum $n_{\mathbf{k}} = \frac{A(t)}{\omega_{\mathbf{k}} + |\mu|}$

Summary of WWT predictions

		Stationay KZ spectra	Self-similar solutions $n^{\rm rad}(k) = 4\pi k^2 n_k$
4-Waves Dimension D=3 $k\xi \gg 1$	Direct energy cascade	$n_k = C_{\rm d} \lvert P_0 \rvert^{1/3} k^{-3} \log^{-\frac{1}{3}}(k/k_{\rm f})$ $C_{\rm d} \approx 5.26 \times 10^{-2}$	$t^{1/2} n^{\rm rad}(k,t) = f(k/t^b)$ $b = \lambda/3 + 1/6\,,$ if $E(t) \sim t^{\lambda}$
	inverse particle cascade	$n_k = C_{\rm i} \lvert Q_O \rvert^{1/3} k^{-7/3}$ $C_{\rm i} = 7.5774045 \times 10^{-2}$	$\tau^{1/2} n^{\rm rad}(k,t) = g(k/\tau^m)\,, \tau = t^* - t$ $m = 1/(2\,x^*),,$ if $n^{\rm rad}(k) \sim k^{-x^*}$

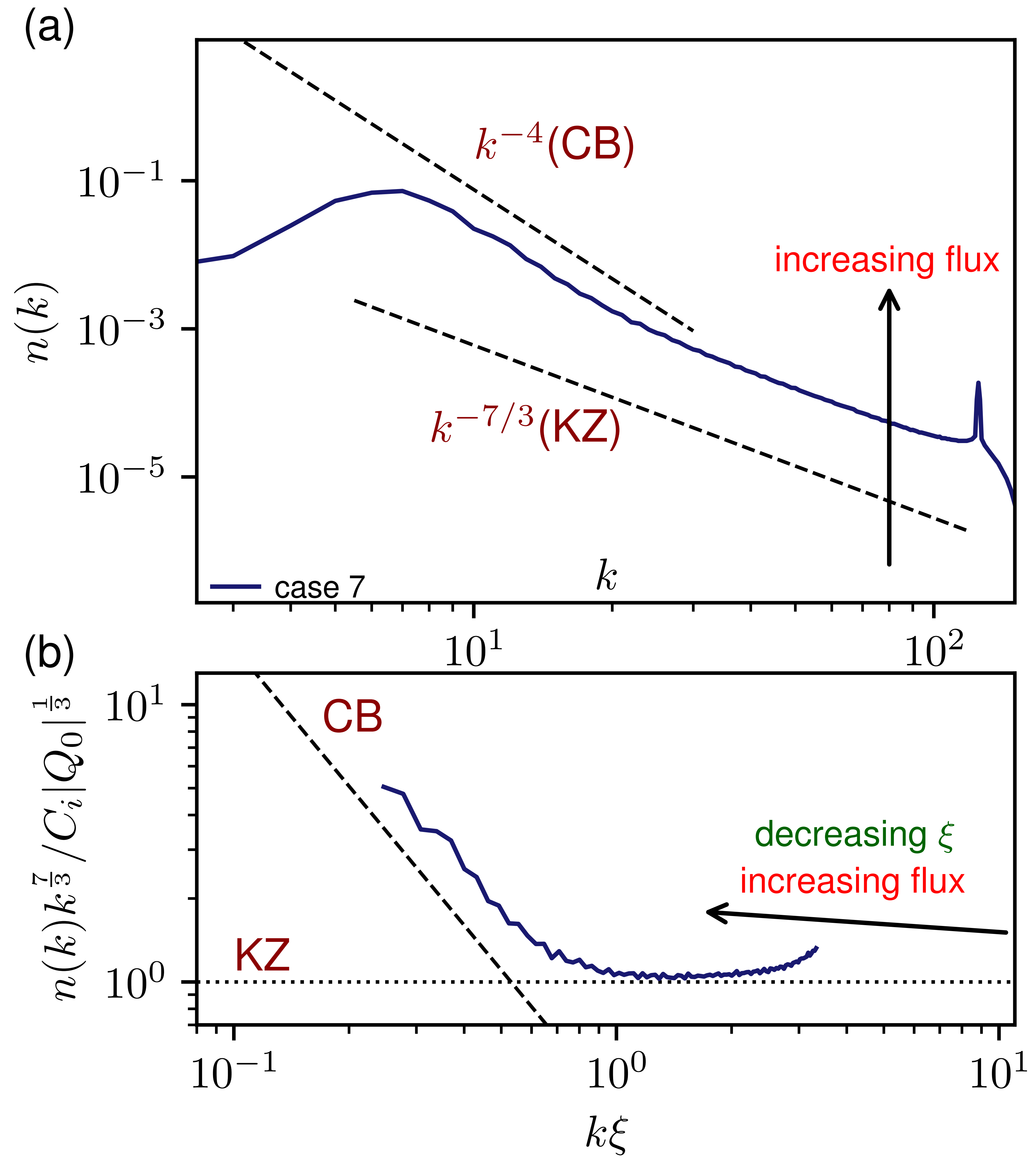
	Acoustic limit $k\xi \ll 1$	Short wave limit $k\xi \gg 1$
3-Waves Dimension D=3	$E_k = C_1 c_s^{1/2} P_0^{1/2} k^{-3/2}$ $\quad n_k \sim k^{-\frac{11}{2}}$ $C_1 = \sqrt{3c_s/(32V_0^2\pi(\pi + 4\ln 2 - 1))}$	$E_k = C_2 c_s^{1/2} \xi^{5/2} P_0^{1/2} k$ $\quad n_k \sim k^{-3}$ $C_2 = 2^{3/4}/\sqrt{\pi(\pi - 4\ln 2)}$
3-Waves Dimension D=2	$E_k = C_{\rm E} c_s^{1/2} a^{1/2} P_O^{1/2} k^{-1}$ $\quad n_k \sim k^{-3}$ $C_{\rm E} = 6^{1/4}\sqrt{c_s}/\pi V_0\,,\quad a = \xi/2$	

Towards Strong Wave Turbulence



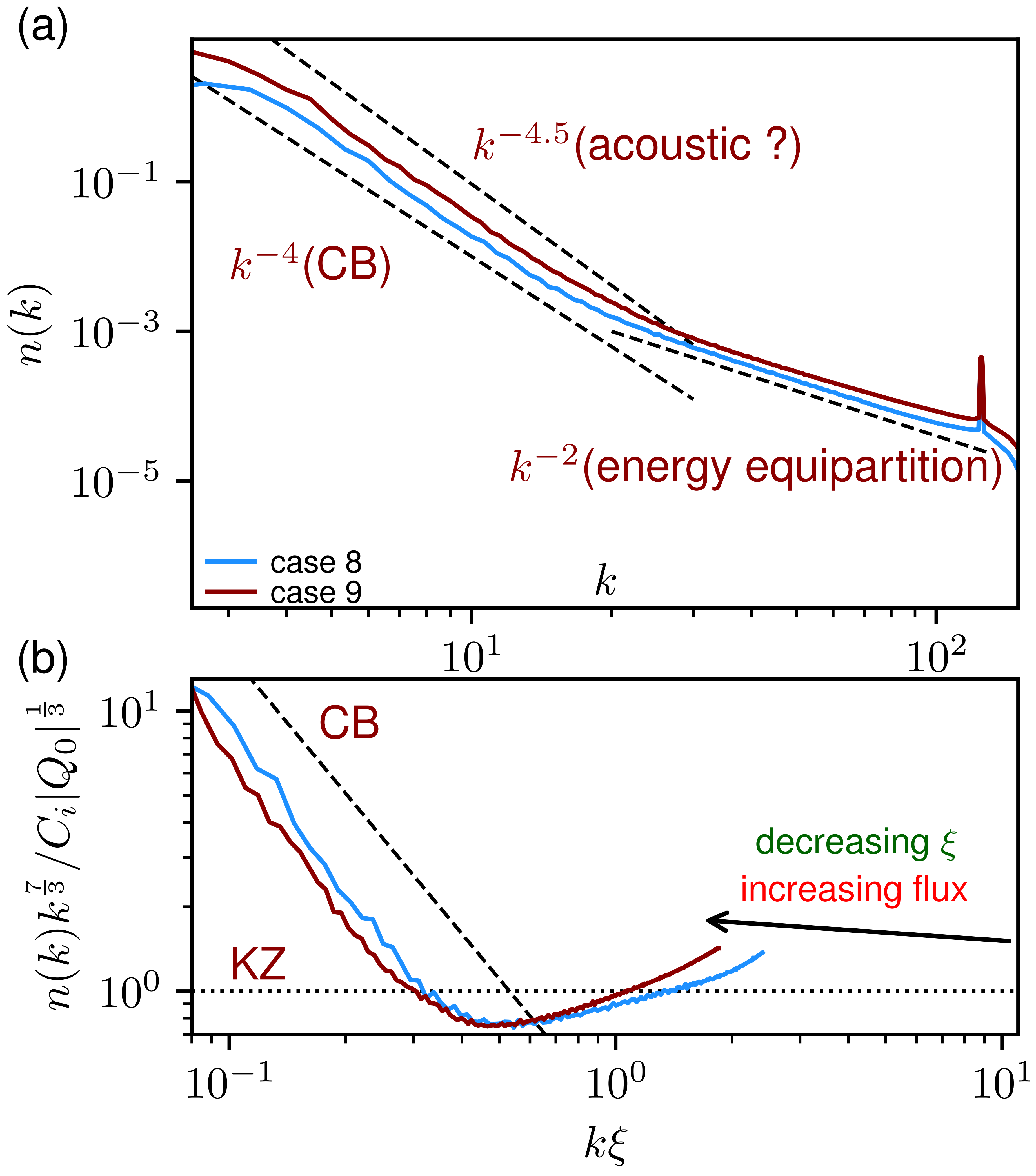
- Steady spectra for the inverse cascade
- KZ scaling survives for 100 times bigger wave amplitude, and 10^6 bigger flux !
- The constant survives as long as KZ scaling occurs for the scales below the healing length

Critical Balance

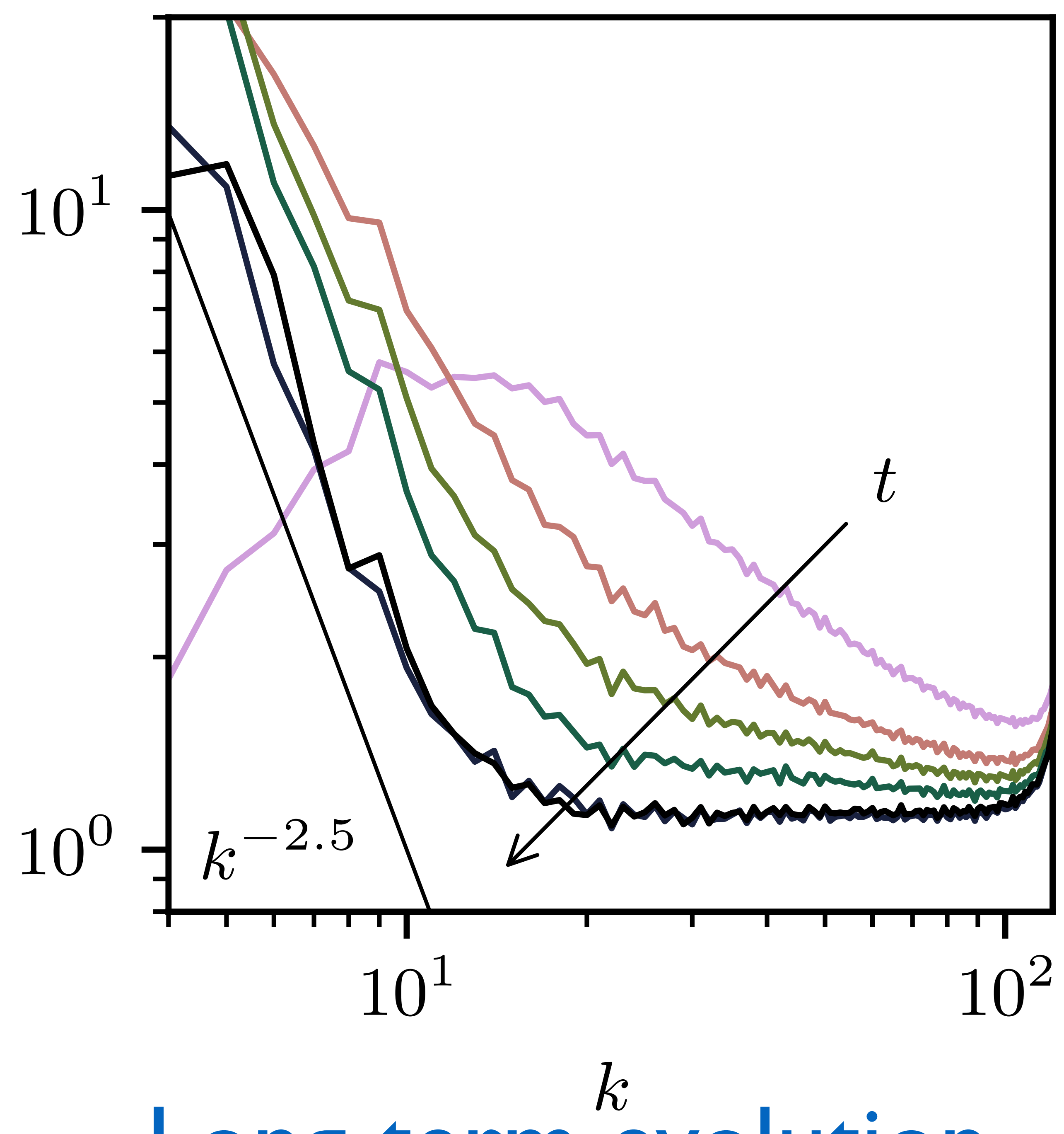


- Critical balance (CB): equating the linear and the nonlinear terms of NLS
- Expected for scales around the healing length
- KZ for small scales

What is this?

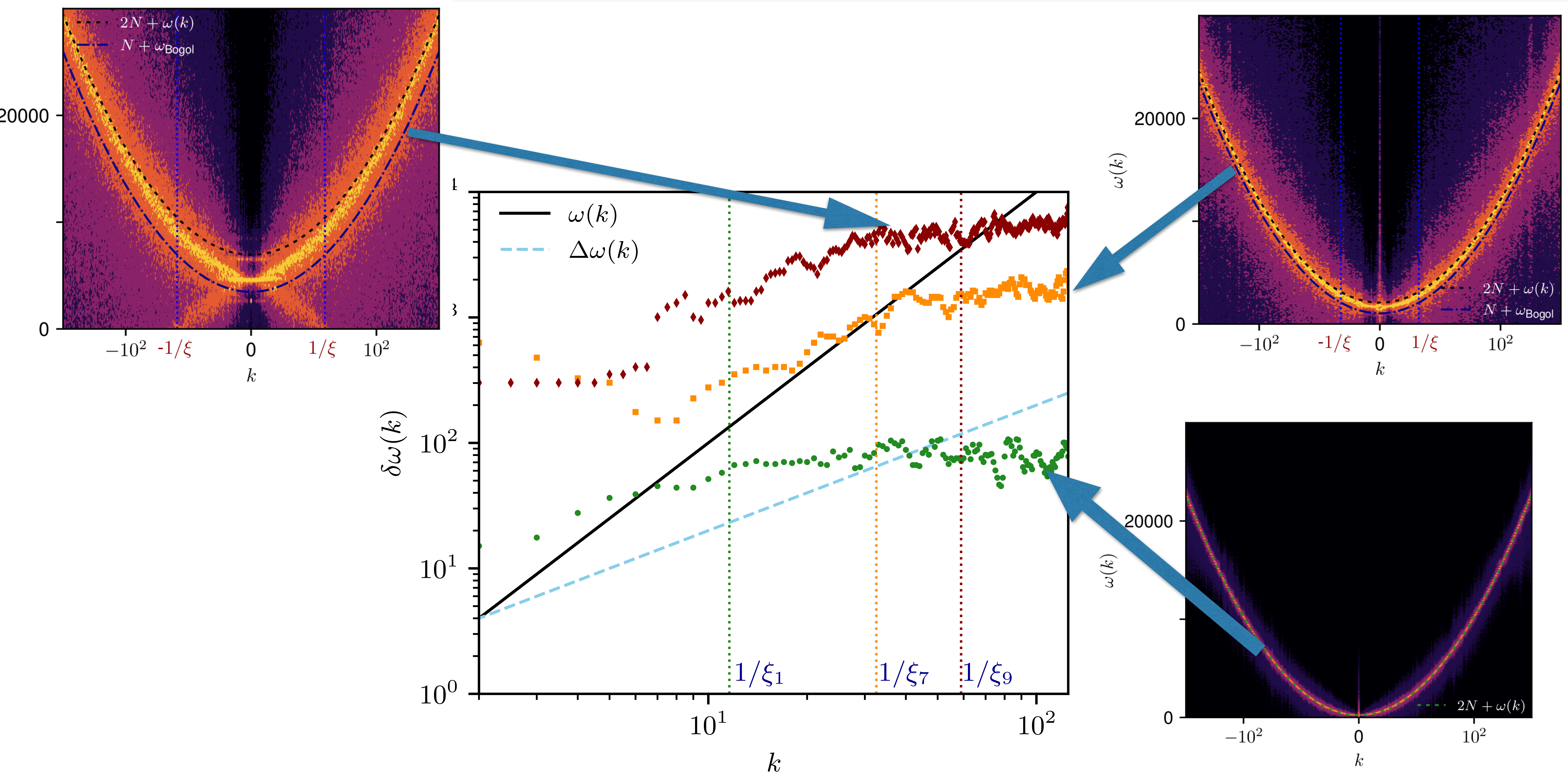


- Is this universal?
- For small k: acoustic?
- For large k: warm cascade?
- Why is it tricky?



Long-term evolution without dissipation at low k

Evidence for Strong Wave Turbulence



Publications

- Griffin A, Krstulovic G, L'vov V S, et al. Energy spectrum of two-dimensional acoustic turbulence. Physical review letters, 2022, 128(22): 224501.
- Zhu Y, Semisalov B, Krstulovic G, et al. Testing wave turbulence theory for the Gross-Pitaevskii system. Physical Review E, 2022, 106(1): 014205. (**Editors's Suggestion**)
- Zhu Y, Semisalov B, Krstulovic G, et al. Direct and inverse cascades in turbulent Bose-Einstein condensates. Physical Review Letters, 2023, 130(13): 133001. (**Cover Story**)
- Zhu Y, Semisalov B, Krstulovic G, et al. Self-similar evolution of wave turbulence in Gross-Pitaevskii system. Physical Review E, 2023, 108(6): 064207.
- Moreno-Armijos, M. A., Fritsch, A. R., García-Orozco, A. D., Sab, S., Telles, G., Zhu, Y., ... & Bagnato, V. S. (2024). Observation of relaxation stages in a non-equilibrium closed quantum system: decaying turbulence in a trapped superfluid. *arXiv preprint arXiv:2407.11237*.
- Zhu Y, Krstulovic G, Nazarenko S. Turbulence and far-from-equilibrium equation of state of Bogoliubov waves in Bose-Einstein Condensates[J]. arXiv preprint arXiv:2408.15163, 2024.
- Zhu Y, Krstulovic G, Sergey Nazarenko. Transition to strong wave turbulence in Bose-Einstein condensates. (In Preparation)

Thank you

спасибо

谢谢